Testing for the Stable Relationship in
High-Dimensional Factor Models*

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Abstract

The principal components estimator of the common factors in high-dimensional
approximate factor models can be inconsistent when there is large temporal instabil-
ity in the factor loadings (Bates et al., 2013). In this paper we test for the stable
factor structure against considerable time variation in the factor loadings in the form
of martingales, which capture the notion of unstable factor relationship that gradu-
ally changes over time. We obtain the asymptotic distribution of the test statistic by

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deriving the conditions under which the estimation error of the common factors is asymptotically negligible for the test statistic. Monte Carlo simulations show that the proposed test performs well and is more powerful than the tests designed for structural breaks to detect unstable factor relationship. We apply the test to a panel of macroeconomic and financial variables in the UK and find the evidence of unstable factor structure during and after the recent financial crisis.

**Keywords:** Factor models, Time-varying factor loadings, LM test

**JEL Classification:** C12, C33

1 Introduction

In the past decade, approximate factor models have been increasingly important in the analysis of high-dimensional macroeconomic datasets, as a small number of common factors can summarize the comovements of a large number of series and the models are found useful in forecasting and policy applications (Bai & Ng, 2008b; Stock & Watson, 2011). Traditionally, factor loadings of each variable are assumed constant over time. Under this assumption, various estimators of the number of factors as well as the principal components estimator of the factors can be consistent within the framework of large observations in both the time and cross-section dimensions. The assumption of loading constancy is however not supported by empirical evidence (Stock & Watson, 2009). Given temporal instability in the factor loadings, the estimators can still be consistent provided that the magnitude of instability is not exceeding some upper bound shown by Bates *et al.* (2013). If time variation is stronger, however, the inference would be misleading and the forecasts
based on the principal components estimator of the factors and the estimator of the number of factors might be affected. Testing for the constancy of factor loadings is therefore necessary before every serious application of factor models. The literature on testing for loading stability has focused on structural breaks, including Breitung & Eickmeier (2011), Yamamoto & Tanaka (2013), Chen et al. (2014), Corradi & Swanson (2014), Cheng et al. (2014) and Han & Inoue (2015). Breitung & Eickmeier (2011) and Yamamoto & Tanaka (2013) tested whether the factor loadings of one specific variable have breaks, while the others concerned the shifts in the factor loadings of all variables.

In this paper, we test against a martingale-type alternative which allows for the gradual shifts in the factor loadings over time. We focus on martingale-type instability rather than structural breaks for four reasons. Firstly, Hansen (1992) argued that specifying the parameter process as a martingale is more appropriate if one is interested in testing whether the model captures a stable relationship. Secondly, it is well known that a factor model with structural breaks in the factor loadings can be equivalently represented by a stable factor model with a larger number of factors. It follows that structural breaks in factor loadings have no theoretical influence on the forecasts based on factor-augmented regressions, if the number of factors is not ex ante imposed but is instead estimated by the information criteria of Bai & Ng (2002), for example. Factor models with martingale-type loadings, however, have no equivalent representations of stable models and the number of factors can not generally be consistently estimated using the existing methods (Takongmo & Stevanovic, 2014). Martingale-type time variation, in this sense, is a more serious issue than structural breaks that would potentially affect the forecasts based on the estimators of the factors. Thirdly, Del Negro & Otrok (2008) estimated a factor model with time-varying
factor loadings using the Bayesian technique. If there is indeed strong instability of the
type we aim to test against, their model might be a better candidate than stable models for
the purpose of forecasting. Finally, it is known that a model with multiple breaks is just a
special case of the models with time-varying parameters in the form of martingales.

Testing parameter stability against the martingale-type alternative was generally ad-
dressed in a likelihood framework in Nyblom (1989), which was extended to the mod-
els with integrated regressors in Hansen (1992) and to panel data models in Yamazaki &
Kurozumi (2014). In this paper, different from the literature on time-varying parameters,
the alternative of time variation in the form of martingales is assumed to start at a known
change point, which is not necessarily at the beginning of the sample. The main challenge
of adapting the general framework for high-dimensional factor models is the accumulation
of estimation error of common factors over time that might change the conventional dis-
tributions of the test statistics. As in Breitung & Eickmeier (2011), in order to obtain the
asymptotic distribution of the test statistic, we derive the conditions on the relative rates of
the number of cross sections and the number of time periods under which the estimation
error of common factors is asymptotically negligible for the test statistic. Generally the
number of cross sections is required to be large enough compared to the size of time peri-
ods after the change point. Our test aims to have nontrivial power against the instabilities
that are not tolerated by the principal components estimator of the factors and the estima-
tors of the number of factors. Simulation studies show that the proposed test has good
performance in finite samples and is more powerful than the tests designed for structural
breaks to detect unstable factor structure. We apply our test to a panel of macroeconomic
and financial data in the UK and find considerable time variation during and after the recent
financial crisis.

The rest of the paper is structured as follows. Section 2 develops the test statistic, introduces the assumptions and derives the asymptotic distributions of the test statistic. Section 3 concerns the model with integrated factors. Section 4 presents the Monte Carlo results of the proposed test, and compares the power of our test with that of the tests designed for structural breaks. We apply the test to a UK dataset in Section 5. Section 6 concludes the paper.

2 The model and test statistics for stationary factors

2.1 The test statistic for the model with a known change point

We study a factor model with individual-specific time-varying factor loadings given by

\[ X_{jt} = \lambda_j^t F_t + u_{jt}, \]

where \( X_{jt} \) is observed for \( j = 1, \ldots, N, \ t = 1, \ldots, T \); \( F_t \) is a \( k \)-dimensional vector of unobserved common factors; \( \lambda_j^t \) is the vector of factor loadings of the \( j \)th variable at time \( t \); and \( u_{jt} \) is the idiosyncratic error. The number of factors \( k \) is unobserved. The factor loadings of a given variable \( i \) can evolve as the random walk after some known change point \( T_a \), while
those of the other variables are constants over time. Specifically,

\[
\lambda_{it} = \begin{cases} 
\lambda_{i1} & t \leq T_a \\
\lambda_{i,t-1} + e_{it} & t > T_a 
\end{cases}
\]

\[
\lambda_{jt} = \lambda_{j1} \quad t = 1, \ldots, T; \ j = 1, \ldots, i - 1, i + 1, \ldots, N.
\]

We assume \(E(e_{it} e_{it}') = h_{NT} \Sigma_e\) where the scalar \(h_{NT}\) indicates the intensity of temporal instability and might depend on \(N\) and \(T\), and \(\Sigma_e\) is known and positive definite. Under this assumption, the null hypothesis is that all factor loadings are constant over time:

\[H_0: h_{NT} = 0,\]

while the alternative hypothesis is that the factor loadings of the \(i\)th variable evolve as a martingale process after a change point:

\[H_1: h_{NT} > 0.\]

Under the null hypothesis, the estimators of the number of factors in Bai & Ng (2002) and the principal components estimator of the factors (Bai, 2003) are consistent under some assumptions as \(N, T \to \infty\). Under the alternative hypothesis, Bates et al. (2013) showed in general \(\sqrt{h_{NT}} = O(1/\min(T, (NT)^{1/2}))\) is needed for consistency of Bai & Ng (2002)’s estimators and \(\sqrt{h_{NT}} = o(T^{-1/2})\) is required for mean square consistency of the principal components estimator. Thus, the desired tests should have non-trivial power against the instabilities which are not tolerated by the estimators, such as \(h_{NT} = a/T\) with \(a\) fixed. Our
alternative hypothesis differs from the one in the literature of time-varying parameters in that \( T_a \) is not necessarily equal to 1. When \( t > T_a \), the model can be written in the vector form as

\[
X_t = \Lambda_1 F_t + v_t + u_t,
\]

where \( X_t = (X_{1t}, \ldots, X_{Nt})' \), \( \Lambda_1 = (\lambda_{11}, \ldots, \lambda_{N1})' \), \( v_t = (0, \ldots, F_t' \sum_{\tau = T_a + 1}^{T_a} e_{i\tau}, \ldots, 0)' \), and \( u_t = (u_{1t}, \ldots, u_{Nt})' \) is independently and identically distributed (i.i.d.) over \( t \) with \( E(u_t u_t') = V \).

We adapt the Lagrange Multiplier (LM) test of Nyblom (1989) for factor models. Define \( \tilde{T} = T - T_a \), \( \mathbf{X} = (X_{T_a+1}, \ldots, X_T)' \), \( \mathbf{F} = (F_{T_a+1}, \ldots, F_T)' \), \( D_F = \text{diag}(I_N \otimes F_{T_a+1}', \ldots, I_N \otimes F_T') \) and \( L \) is a \( \tilde{T} \times \tilde{T} \) lower triangular matrix with all non-zero elements equal to 1. Let

\[
\Sigma = I_{\tilde{T}} \otimes V + h_{NT} D_F (L \otimes I_{kN}) (I_{\tilde{T}} \otimes \Sigma_e) (L' \otimes I_{kN}) D_F'
\]

denote the conditional covariance matrix of \( \text{vec}(X') \) given \( F \) where \( \Sigma_e \) is a \( kN \times kN \) sparse matrix with the only non-zero \( k \times k \) block from the \( ik + 1 \) th element to the \( (i + 1)k \) th element on the diagonal being equal to \( \Sigma_e \). Under normality, the joint log-likelihood of \( (X'_{T_a+1}, \ldots, X'_T) \) given \( (F_{T_a+1}', \ldots, F'_T) \) is

\[
l_{\text{vec}(X')|E}(\Lambda_1, h_{NT}, V, \Sigma_e) = -N_{\tilde{T}} \log(2\pi) - \frac{1}{2} \log|\Sigma(h_{NT})| - \frac{1}{2} [\text{vec}(X') - \text{vec}(\Lambda_1 F')]' \Sigma_e^{-1} [\text{vec}(X') - \text{vec}(\Lambda_1 F')].
\]

The maximum likelihood estimator of \( \Lambda_1 \) under the null hypothesis \( h_{NT} = 0 \) is \( \hat{\Lambda}_1 = \)
Let
\[ f(X_t|F_t; \hat{\Lambda}_1, V) = -\frac{N}{2} \log(2\pi) - \frac{1}{2} \log|V| - \frac{1}{2} [X_t - \hat{\Lambda}_1 F_t]' V^{-1} [X_t - \hat{\Lambda}_1 F_t] \]
denote the marginal density of \( X_t \) given \( F_t \) under the null and let \( C_i \) be a \( N \times N \) diagonal matrix with the \( i \)th diagonal element being 1 and the others being 0. Define \( G = C_i \otimes \Sigma_e = (C_i \otimes \Sigma_{ee})(C_i \otimes \Sigma'_{ee}) \) where \( \Sigma_e = \Sigma_{ee} \Sigma'_{ee} \) is the Cholesky decomposition of \( \Sigma_e \). The self-normalized statistic adapted from Nyblom (1989) can be expressed as

\[
LM_i = \frac{1}{T^2 V_{ii}^{-1}} \sum_{\tau=1}^T \sum_{t=T_a+\tau}^T \frac{\partial f(X_t|F_t)}{\partial \text{vec}(\hat{\Lambda}_1')} G \sum_{t=T_a+\tau}^T \frac{\partial f(X_t|F_t)}{\partial \text{vec}(\hat{\Lambda}_1')},
\]

where \( V_{ii}^{-1} \) denote the \( i \)th diagonal element of \( V^{-1} \). Let \( V_{i,-1} \) denote the \( i \)th row of \( V^{-1} \). It follows that

\[
\frac{\partial f(X_t|F_t)}{\partial \text{vec}(\hat{\Lambda}_1')} = \text{vec}(F_t X_t' V^{-1}) - \text{vec}(F_t F_t'(E'E)^{-1}E'XV^{-1}),
\]

and

\[
\sum_{t=T_a+\tau}^T (C_i \otimes \Sigma'_{ee}) \frac{\partial f(X_t|F_t)}{\partial \text{vec}(\hat{\Lambda}_1')} = \Sigma'_{ee} [ \sum_{t=T_a+\tau}^T F_t V_{i,-1} X_t - \sum_{t=T_a+\tau}^T F_t F_t'( \sum_{t=T_a+1}^T F_t F_t')^{-1} \sum_{t=T_a+1}^T F_t V_{i,-1} X_t].
\]

From these results, we obtain
\[ \text{LM}_i = \frac{1}{T^2 V_{ii}} \sum_{\tau=1}^{T} S'_{i\tau} \Sigma e S_{i\tau}, \]

where

\[ s_{i\tau} = \sum_{t=T_{a}+1}^{T_{a}+\tau} F_i V_i^{-1} (u_t + v_t) - \sum_{t=T_{a}+1}^{T_{a}+\tau} F_i F'_i (\sum_{t=T_{a}+1}^{T} F_i F'_i)^{-1} \sum_{t=T_{a}+1}^{T} F_i V_i^{-1} (u_t + v_t) \]

\[ = \sum_{t=T_{a}+1}^{T_{a}+\tau} F_i V_i^{-1} X_t - \sum_{t=T_{a}+1}^{T_{a}+\tau} F_i F'_i (\sum_{t=T_{a}+1}^{T} F_i F'_i)^{-1} \sum_{t=T_{a}+1}^{T} F_i V_i^{-1} X_t. \]

Note that \( \text{LM}_i \) contains unobserved common factors and unknown model parameters including the number of factors \( k \) and the covariance matrix \( V \). Given a consistent estimator of the number of factors, \( \hat{k} \), the principal components estimator of the factors \( \hat{F} \) is \( \sqrt{T} \) times the eigenvectors corresponding to the \( \hat{k} \) largest eigenvalues of \( XX' \) where \( X = (X_1, \ldots, X_T)' \). Let \( \hat{V}_{i:}^{-1} \) denote the \( i \)th row of \( \hat{V}^{-1} \), where \( \hat{V} \) is a consistent estimator of \( V \). After replacing \( \Sigma_e \) with \( \hat{\Sigma}_F^{-1} \) where \( \hat{\Sigma}_F = \frac{1}{T} \sum_{t=1}^{T} \hat{F}_t \hat{F}'_t \) in order to remove the nuisance parameters in the asymptotic distribution (see more discussions in Section 2.2), the computable test statistic \( \hat{\text{LM}}_i \) can be written as

\[ \hat{\text{LM}}_i = \frac{1}{T^2 \hat{V}_{ii}^{-1}} \sum_{\tau=1}^{T} \hat{S}'_{i\tau} \hat{\Sigma}_F^{-1} \hat{S}_{i\tau}, \]

where

\[ \hat{s}_{i\tau} = \sum_{t=T_{a}+1}^{T_{a}+\tau} \hat{F}_i \hat{V}_i^{-1} X_t - \sum_{t=T_{a}+1}^{T_{a}+\tau} \hat{F}_i \hat{F}'_i (\sum_{t=T_{a}+1}^{T} \hat{F}_i \hat{F}'_i)^{-1} \sum_{t=T_{a}+1}^{T} \hat{F}_i \hat{V}_i^{-1} X_t. \]
2.2 Asymptotics for the test statistic

In order to derive the asymptotic distribution of the test statistic, we make the following assumptions on common factors, factor loadings and idiosyncratic errors. Let \( \| A \| = \sqrt{\text{tr}(A'A)} \) be the Frobenius norm, and let \( \lambda_{\text{min}}(A) \) and \( \lambda_{\text{max}}(A) \) denote the minimum and maximum eigenvalues of a symmetric matrix \( A \), respectively.

**Assumptions**

1. (Common factors) \( \{F_t\} \) is stationary and ergodic, \( \mathbb{E}(F_tF'_t) = \Sigma_F \) for some positive definite \( \Sigma_F \) and \( \mathbb{E}(\| F_t \|^4) < \infty \) for \( t = 1, \ldots, T \).

2. (Initial factor loadings) \( \| \lambda_{j1} \| \leq \overline{\lambda} < \infty \) for \( j = 1, \ldots, N \) and \( \| \Lambda'_1 \Lambda_1/N - \Sigma_\Lambda \| \to 0 \) as \( N \to \infty \) for some positive definite \( \Sigma_\Lambda \).

3. (Factor loading innovations of the \( i \)th variable)

   (a) \( e_{it} \) is i.i.d. across \( t \), \( \mathbb{E}(e_{it}) = 0 \) and \( \mathbb{E}(e_{it}e'_{it}) = h_{NT}\Sigma_e \) where \( \Sigma_e \) is known and positive definite.

   (b) \( e_{it} \) is independent of both \( F_s \) and \( u_{js} \) for all \( t, s, j \).

4. (Idiosyncratic errors) \( M < \infty \) is a positive constant. For all \( N, T \) and \( \bar{T} \),

   (a) \( u_t \) is i.i.d., \( \mathbb{E}(u_t) = 0 \), \( \mathbb{E}(u_tu'_t) = V \) and \( \mathbb{E}(u_{jt}^8) < \infty \) for \( j = 1, \ldots, N \).

   (b) \( \max_j \sum_{k=1}^N |V_{kj}| \leq M \) for all \( N \).

   (c) \( \mathbb{E}|N^{-1/2} \sum_{j=1}^N [u_{js}u_{jt} - \mathbb{E}(u_{js}u_{jt})]|^4 \leq M \) for every \( s \) and \( t \).

   (d) For every \( t \), \( \mathbb{E} \left( \left| \frac{1}{\sqrt{NT}} \sum_{s=1}^T \sum_{j=1}^N F_s [u_{js}u_{jt} - \mathbb{E}(u_{js}u_{jt})] \right|^2 \right) \leq M. \)
(e) For every \( t \), \( \mathbb{E} \left| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \lambda_j u_{jt} \right|^4 \leq M \).

(f) \( u_{jt} \) and \( F_s \) are independent for all \( j, t, s \).

(g) Let \( V_{ij}^{-1} \) denote the \( ij \)th element of \( V^{-1} \).

i. The number of non-zero elements in \( V_{i}^{-1} \) is \( r_{i,N} \) which is finite when \( N \to \infty \).

ii. The number of non-zero elements in \( \hat{V}_{i}^{-1} \) is \( \hat{r}_{i,N} \) which is finite almost surely when \( N \to \infty \).

iii. \( \hat{V}_{ij}^{-1} \overset{P}{\to} V_{ij}^{-1} \) for each \( i, j, N \).

(h) \( \frac{1}{T} \sum_{t=1}^{T} \left| \frac{1}{\sqrt{NT}} \sum_{s=1}^{T} \sum_{j=1}^{N} \lambda_j (u_{js} - \mathbb{E}(u_{js})) \right|^2 = O_p(1) \) for each \( i \).

**Comments on the assumptions:**

**Part of Assumption 1, Assumptions 2 and 4(b)-(e):** These assumptions are borrowed from Bai (2003) and are aimed at consistent estimation of the common factors and the number of factors under the null hypothesis. Generally idiosyncratic errors are allowed to be weakly dependent and heteroskedastic in the cross-section dimension.

**Assumption 4(a):** The assumption of i.i.d. idiosyncratic errors in the time dimension is made for convenience only. It can be weakened to allow for limited time serial dependence and heteroskedasticity.

**Assumption 4(gi):** Examples include cross-section heteroskedasticity with no dependence and cross-section heteroskedasticity with block-diagonal correlation structure of fixed block size (Choi, 2012), among others. The block-diagonal structure is motivated by the fact that in macroeconomic or financial applications the idiosyncratic components often represent industry-specific shocks and are almost uncorrelated among the variables.
across different industries (Bai & Liao, 2013).

**Assumptions 4(gii) and 4(giii):** Let \( \hat{u}_t \) denote the residual vector from the principal components estimation. The sample error covariance matrix \( \hat{V} = \frac{1}{T} \sum_{t=1}^{T} \hat{u}_t \hat{u}_t' \) can satisfy these conditions provided we have the prior knowledge on the structure of the idiosyncratic covariance matrix (e.g. block diagonal)(Choi, 2012).

**Assumptions 4(h):** This condition is not so strict as all sums include variables with zero means.

Under the null hypothesis and Assumptions 1 and 4, as \( \bar{T} \to \infty \),

\[
\frac{1}{\sqrt{\bar{T}} V^{-1}_{ii}} \sum_{i=1}^{T} s_{i[T]} \to D \sum_{i}^{1/2} (W(r) - r W(1)),
\]

and

\[
LM_i \to D \int_0^1 (W(r) - r W(1))^T \sum_{i}^{1/2} \sum_{i}^{1/2} (W(r) - r W(1)) dr.
\]

As is well known, one can only consistently estimate the \( k \)–dimensional space spanned by the true factors. Let \( V_{NT} \) denote the \( k \times k \) diagonal matrix of the first \( k \) largest eigenvalues of \( XX'/TN \). The principal components estimator \( \hat{F} \) is a consistent estimator of the transformed factors \( \hat{F} = FH \) where \( F = (F_1, \ldots, F_T)' \) and \( H = (\Lambda_1' \Lambda_1/N)(F' \hat{F} / T)V_{NT}^{-1} \) (Bai, 2003). Rewriting \( LM_i \) by replacing \( F_i \) with \( \hat{F} \) yields \( LM_i = \frac{1}{T \Sigma e_{i}} \sum_{\tau=1}^{T} \hat{s}_{i}^{T} \Sigma e \hat{s}_{i} \) where

\[
\hat{s}_{i} = \sum_{t=T_{a}+1}^{T_{a}+\tau} \hat{F}_{i} V_{i}^{-1} X_t - \sum_{t=T_{a}+1}^{T_{a}+\tau} \hat{F}_{i} \hat{F}_{i}^{'} ( \sum_{t=T_{a}+1}^{T} \hat{F}_{i} \hat{F}_{i}^{'} )^{-1} \sum_{t=T_{a}+1}^{T} \hat{F}_{i} V_{i}^{-1} X_t.
\]
As \( N, \tilde{T} \to \infty \),

\[
\text{LM}_i \xrightarrow{D} \int_0^1 (W(r) - rW(1))' \Sigma_F^{1/2} \hat{H} \Sigma_e \hat{H}' \Sigma_F^{1/2} (W(r) - rW(1)) \, dr,
\]

where \( \hat{H} = \text{plim}_{N,T \to \infty} H \). The limit distribution of \( \text{LM}_i \) can be free from nuisance parameters if \( \Sigma_e \) is replaced with \( \tilde{\Sigma}^{-1} \) where \( \tilde{\Sigma} = \hat{H}' \Sigma_F \hat{H} = \text{plim}_{N,T \to \infty} (\frac{1}{T} \sum_{t=1}^T \tilde{F}_t \tilde{F}_t') \). After replacement, the new statistic is indicated by \( \tilde{\text{LM}}_i = \frac{1}{\tilde{T}^{1/2}v_{ii}} \sum_{t=1}^{\tilde{T}} \tilde{v}_t' \tilde{\Sigma}^{-1} \tilde{v}_t \). As \( N, \tilde{T} \to \infty \),

\[
\tilde{\text{LM}}_i \xrightarrow{D} \int_0^1 (W(r) - rW(1))'(W(r) - rW(1)) \, dr,
\]

which is the generalized Von Mises distribution with \( k \) degrees of freedom. We are interested in the condition on the relative rates of \( N, T, \tilde{T} \) under which the estimation error of factors and related variables is negligible for the asymptotic distribution of \( \tilde{\text{LM}}_i \).

**Theorem 1:** Under the null hypothesis and Assumptions 1-4, if \( \frac{\sqrt{T}}{\min(N^2,NT,T^2)} = o(1) \), then \( \tilde{\text{LM}}_i = \text{LM}_i + o_p(1) \) as \( N, \tilde{T} \to \infty \). Thus, \( \tilde{\text{LM}}_i \xrightarrow{D} \int_0^1 (W(r) - rW(1))'(W(r) - rW(1)) \, dr \).

**Comments on Theorem 1:** If \( \tilde{T} = T \), the condition reduces to \( \frac{\sqrt{T}}{\min(N^2,T^2)} = o(1) \), which is same as the condition for the break tests in Breitung & Eickmeier (2011) if \( N \leq T \). Generally, \( N \) is required to be large enough for the estimator error of factors to become asymptotically negligible.

Next, we investigate the asymptotic distribution of the test statistic under the local alternative \( h_{NT} = a/\tilde{T}^2 \) with \( a \) fixed. Under the assumptions, as \( \tilde{T} \to \infty \),

\[
\frac{1}{\sqrt{T}} \sum_{t=T_{a+1}}^{T_{a+[\tilde{T}]}-1} F_t V_t^{-1} v_t \xrightarrow{D}
\]

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Table 1: Critical values of the generalized Von Mises distribution with $k$ degrees of freedom (Source: Hansen (1990), Table 1)

<table>
<thead>
<tr>
<th>Significance Level</th>
<th>1%</th>
<th>5%</th>
<th>10%</th>
</tr>
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<tr>
<td>$k$</td>
<td>1%</td>
<td>5%</td>
<td>10%</td>
</tr>
<tr>
<td>1</td>
<td>0.748</td>
<td>0.470</td>
<td>0.353</td>
</tr>
<tr>
<td>2</td>
<td>1.070</td>
<td>0.749</td>
<td>0.610</td>
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<tr>
<td>3</td>
<td>1.350</td>
<td>1.010</td>
<td>0.846</td>
</tr>
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<td>4</td>
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<td>1.880</td>
<td>1.470</td>
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<tr>
<td>10</td>
<td>3.050</td>
<td>2.540</td>
<td>2.290</td>
</tr>
</tbody>
</table>

$\sqrt{V_{ii}^{-1} \Sigma \epsilon^{1/2} W_2(r)}$ where $W_2(r) = \int_0^r W(s) ds$. It follows that

$$\hat{\text{LM}}_i \xrightarrow{D} \frac{1}{V_{ii}^{-1}} \int_0^1 W_3'(r) \Sigma \epsilon W_3(r) dr,$$

where $W_3(r) = \sqrt{V_{ii}^{-1} \Sigma \epsilon^{1/2}} (W_2(r) - rW_2(1)) + \sqrt{V_{ii}^{-1} \Sigma \epsilon^{1/2}} (W(r) - rW(1))$. Furthermore,

$$\hat{\text{LM}}_i \xrightarrow{D} \frac{1}{V_{ii}^{-1}} \int_0^1 W_3'(r) \Sigma \epsilon^{-1} W_3(r) dr.$$

Note that the alternative distribution depends on $V_{ii}, \Sigma \epsilon$ and $\Sigma \epsilon$. The condition under which the estimation error of factors is insignificant for the asymptotic alternative distribution of the test statistic is same as that stated in Theorem 1.

**Theorem 2:** Given Assumptions 1-4, under the local alternative $h_{NT} = a/\tilde{T}^2$ with $a$ fixed, if $\frac{\sqrt{T}}{\min(N^2, NT, T^2)} = o(1)$, then $\hat{\text{LM}}_i = \text{LM}_i + o_p(1)$ as $N, \tilde{T} \to \infty$. Therefore we have

$$\hat{\text{LM}}_i \xrightarrow{D} \frac{1}{V_{ii}^{-1}} \int_0^1 W_3'(r) \Sigma \epsilon^{-1} W_3(r) dr.$$
3 Model and the test statistic for integrated factors

In this section, common factors are assumed to be integrated of order one with no cointegration among them. Specifically, $F_t = F_{t-1} + \epsilon_t$ for $t = 1, \ldots, T$ where $E(\epsilon_t \epsilon_t') = \Sigma_e$ and $\epsilon_t$ is independent of $\epsilon_{is}$ and $u_{js}$ for all $t, s, j$. As a result of integration, Assumption 4(d) does not hold any longer. Now the statistic is defined as a function of the relative change point $\pi$:

$$\text{LM}_i(\pi) = \frac{1}{T^2 \hat{V}_{ii}^{-1}} \sum_{\tau=1}^{T-[T\pi]} s_{i\tau}(\pi) \Sigma_e s_{it}(\pi),$$

where $s_{i\tau}(\pi) = \sum_{t=[T\pi]+1}^{[T\tau]+1} F_{t} V_{i}^{-1} X_t - \sum_{t=[T\pi]+1}^{[T\tau]+1} F_{t} F_{t}' \left( \Sigma_{t=[T\pi]+1}^{T} F_{t} F_{t}' \right)^{-1} \sum_{t=[T\pi]+1}^{T} F_{t} V_{i}^{-1} X_t$.

Under Assumptions 4, $\frac{1}{T^2} \sum_{t=[T\pi]+1}^{[T\tau]+1} F_{t} F_{t}' \Sigma_e^{1/2} \int_{\pi}^{r\pi/T+\pi} W(\tilde{r}) W'(\tilde{r}) d\tilde{r} \Sigma_e^{1/2}$ and $\frac{1}{T} \sum_{t=[T\pi]+1}^{[T\tau]+1} F_{t} V_{i}^{-1} u_t \overset{D}{\rightarrow} \int_{\pi}^{r\pi/T+\pi} V_{ii}^{-1} \Sigma_e^{1/2} W(\tilde{r}) dW_4(\tilde{r})$ where $W_4(\cdot)$ stands for a standard Brownian motion and is independent of $W(\cdot)$, so

$$\frac{1}{T} \sqrt{V_{ii}^{-1}} s_{i[T\tau]}(\pi) \overset{D}{\rightarrow} \Sigma_e^{1/2} \tilde{W}(r, \pi),$$

where

$$\tilde{W}(r, \pi) = \int_{\pi}^{r\pi/T+\pi} W(\tilde{r}) dW_4(\tilde{r})$$

$$- \int_{\pi}^{r\pi/T+\pi} W(\tilde{r}) W'(\tilde{r}) d\tilde{r} \left( \int_{\pi}^{1} W(\tilde{r}) W'(\tilde{r}) d\tilde{r} \right)^{-1} \int_{\pi}^{1} W(\tilde{r}) dW_4(\tilde{r}).$$

Consequently,

$$\text{LM}_i(\pi) \overset{D}{\rightarrow} \int_{0}^{1} \tilde{W}(r, \pi)' \Sigma_e^{1/2} \Sigma_e^{1/2} \tilde{W}(r, \pi) dr.$$
eigenvectors corresponding to the $k$ largest eigenvalues of the matrix $XX'$, as in Bai (2004).

Let $V_{NT}$ denote the $k \times k$ diagonal matrix of the first $k$ largest eigenvalues of $XX'/(T^2N)$.

As in the case of stationary factors, one can transform the factors as $\tilde{F}_{I,t} = H_I^t F_I$ where $H_I = (\Lambda' \Lambda_1/N)(F' \hat{F}_I/T^2) V_{NT}^{-1}$. We define $\hat{L}_{M_i}(\pi) = \frac{1}{TT' \hat{V}^{-1}_ii} \sum_i^{T-\lfloor T \pi \rfloor} \hat{s}_i(\tau)(\hat{s}_i-1)\hat{s}_i(\tau)$ where $\hat{s}_i(\tau) = \sum_{t=\lfloor T \pi \rfloor+1}^{\lfloor T \pi \rfloor+\tau} \tilde{F}_{I,t} V_{i,-}^{-1} X_t - \sum_{t=\lfloor T \pi \rfloor+1}^{\lfloor T \pi \rfloor+\tau} \tilde{F}_{I,t} \tilde{F}_I (\sum_{t=\lfloor T \pi \rfloor+1}^{\tau} \tilde{F}_{I,t} \tilde{F}_I - 1) \sum_{t=\lfloor T \pi \rfloor+1}^{\tau} \tilde{F}_{I,t} V_{i,-}^{-1} X_t$ and

$$\hat{L}_{M_i}(\pi) \overset{D}{\rightarrow} \int_0^1 \tilde{W}(r,\pi)'\tilde{W}(r,\pi)dr.$$ 

The distribution of $\int_0^1 \tilde{W}(r,\pi)'\tilde{W}(r,\pi)dr$ depends on the relative change point $\pi$ and the dimension $k$. The critical values listed in Table 2 are obtained by simulating the distribution of $\int_0^1 \tilde{W}(r,\pi)'\tilde{W}(r,\pi)dr$ for the dimension $k$ from 1 to 5 and a selection of $\pi$ from 0.4 to 0.8.

We obtain a single realization from the distribution by simulating a $k$-dimensional Brownian motion and a standard Brownian motion which are independent of each other. We compute the quantiles from 500 realizations and then average them over 100 repetitions.

As for the case of stationary factors, we estimate $\hat{L}_{M_i}(\pi)$ by $\hat{L}_{M_i}(\pi)$ such that

$$\hat{L}_{M_i}(\pi) = \frac{1}{TT' \hat{V}^{-1}_ii} \sum_i^{T-\lfloor T \pi \rfloor} \hat{s}_i(\tau)(\hat{s}_i-1)\hat{s}_i(\tau),$$

where $\hat{s}_i(\tau) = \sum_{t=\lfloor T \pi \rfloor+1}^{\lfloor T \pi \rfloor+\tau} \tilde{F}_{I,t} V_{i,-}^{-1} X_t - \sum_{t=\lfloor T \pi \rfloor+1}^{\lfloor T \pi \rfloor+\tau} \tilde{F}_{I,t} \tilde{F}_I (\sum_{t=\lfloor T \pi \rfloor+1}^{\tau} \tilde{F}_{I,t} \tilde{F}_I - 1) \sum_{t=\lfloor T \pi \rfloor+1}^{\tau} \tilde{F}_{I,t} V_{i,-}^{-1} X_t$ and $\hat{S}_e = \frac{1}{\tau-1} \sum_{t=1}^{\tau-1} (\hat{F}_{I,t+1} - \hat{F}_{I,t}')(\hat{F}_{I,t+1} - \hat{F}_{I,t})'$.

**Theorem 3:** Under the null hypothesis, $\hat{L}_{M_i}(\pi) = \hat{L}_{M_i}(\pi) + O_p\left(\frac{1}{\sqrt{\min(N,T^2)}}\right)$ for any $\pi$ as $N,T \rightarrow \infty$. Thus, $\hat{L}_{M_i}(\pi) \overset{D}{\rightarrow} \int_0^1 \tilde{W}(r,\pi)'\tilde{W}(r,\pi)dr$.
Table 2: Critical values of the distribution $\int_0^1 \tilde{W}(r, \pi) / \tilde{W}(r, \pi) dr$ by simulation.

<table>
<thead>
<tr>
<th>$k$ = 1</th>
<th>$k$ = 2</th>
<th>$k$ = 3</th>
<th>$k$ = 4</th>
<th>$k$ = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>1%</td>
<td>5%</td>
<td>10%</td>
<td>1%</td>
</tr>
<tr>
<td>0.4</td>
<td>0.559</td>
<td>0.242</td>
<td>0.149</td>
<td>0.644</td>
</tr>
<tr>
<td>0.45</td>
<td>0.554</td>
<td>0.234</td>
<td>0.143</td>
<td>0.628</td>
</tr>
<tr>
<td>0.5</td>
<td>0.532</td>
<td>0.224</td>
<td>0.139</td>
<td>0.611</td>
</tr>
<tr>
<td>0.55</td>
<td>0.494</td>
<td>0.216</td>
<td>0.129</td>
<td>0.558</td>
</tr>
<tr>
<td>0.6</td>
<td>0.476</td>
<td>0.204</td>
<td>0.123</td>
<td>0.526</td>
</tr>
<tr>
<td>0.65</td>
<td>0.429</td>
<td>0.185</td>
<td>0.113</td>
<td>0.488</td>
</tr>
<tr>
<td>0.7</td>
<td>0.384</td>
<td>0.166</td>
<td>0.100</td>
<td>0.426</td>
</tr>
<tr>
<td>0.75</td>
<td>0.336</td>
<td>0.145</td>
<td>0.089</td>
<td>0.379</td>
</tr>
<tr>
<td>0.8</td>
<td>0.285</td>
<td>0.123</td>
<td>0.074</td>
<td>0.316</td>
</tr>
</tbody>
</table>

4 Monte Carlo Results

The Monte Carlo results are presented in this section. We consider a two-factor model with $N = 50, 100, 150, 200$ and $T = 100, 150, 200, 250$, which are typical in empirical applications. $\{F_t\}$ is either i.i.d. $N(0, I_2)$ or random walk with $F_0 = [0, 0]^T$ and $N(0, I_2)$ error terms. In all cases, we estimate the number of factors using Onatski (2010)’s edge distribution estimator. That is, $\hat{k} = \max \{k \leq k_{\text{max}}, \sigma_k - \sigma_{k+1} \geq c\}$ where $k_{\text{max}}$ is a given upper bound (set to 10 in our study), $\sigma_k$ is the $k$th largest eigenvalue of $X'X/T$ and $c$ is a threshold value estimated from the empirical distribution of eigenvalues. The idiosyncratic structure is block diagonal, and the size of each block is 5 (Choi, 2012). We generate each block by i.i.d. normal distribution. The idiosyncratic covariance matrix is estimated by the sample error covariance matrix $\hat{V} = \frac{1}{T} \sum_{t=1}^{T} \hat{u}_t \hat{u}_t'$. It is a consistent estimator under the assumption of prior knowledge of the block diagonal structure.
Factor loadings for the variable $i$ are generated by the random walk

$$
\lambda_{it} = \begin{cases} 
\lambda_{i1} & t \leq T_a \\
\lambda_{i,t-1} + e_{it} & t > T_a,
\end{cases}
$$

where $\lambda_{i1}$ is i.i.d. $N(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, I_2)$; $E(e_{it}e_{it}') = \frac{a}{T} \begin{bmatrix} 1 & \sigma_{\lambda}^2 \\
\sigma_{\lambda}^2 & 1 \end{bmatrix}$ such that $a = 0, 1, 5, 10, 20$ and $\sigma_{\lambda}^2$ follows $\text{Uniform}(0, 0.9)$; $T_a$ is set equal to $\frac{T}{2}$.

We use 5000 replications for Monte Carlo simulation. Table 2 shows the results of empirical sizes. When common factors are stationary, the test tends to be oversized, due to the finite sample bias of the estimator of the number of factors. If the number of factors is set equal to the true value, the empirical sizes would become closer to the nominal size of 0.05. In the case of integrated factors, the rejection frequencies turn out to be close to the nominal size. Tables 3 and 4 report the results on the power of the tests for the models with stationary factors and integrated factors respectively. In both cases, it seems that the tests are consistent in that the power grows as $N$ and $T$ increase. When factors are integrated, the test suffers from the issue of nonmonotonic power, which is not unusual in the tests for structural stability.

The tests designed for structural breaks, such as likelihood ratio (LR), Wald and Lagrange multiplier statistics in Breitung & Eickmeier (2011), also have power for martingale-type time variation. In Figure 1, we compare our LM statistic with them, which assume

---

1. $i$ is fixed in each experiment and is taken across $1, \ldots, N$. 
Table 3: Empirical sizes (nominal size is 0.05)

<table>
<thead>
<tr>
<th></th>
<th>I(0) Factors</th>
<th></th>
<th>I(1) Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T = 100$</td>
<td>$T = 150$</td>
<td>$T = 200$</td>
</tr>
<tr>
<td>$N = 50$</td>
<td>0.069</td>
<td>0.077</td>
<td>0.081</td>
</tr>
<tr>
<td>$N = 100$</td>
<td>0.067</td>
<td>0.071</td>
<td>0.082</td>
</tr>
<tr>
<td>$N = 150$</td>
<td>0.070</td>
<td>0.068</td>
<td>0.071</td>
</tr>
<tr>
<td>$N = 200$</td>
<td>0.063</td>
<td>0.070</td>
<td>0.071</td>
</tr>
</tbody>
</table>

there is a break at $T_d$. The result shows that the performance of the tests for breaks is inferior to that of LM for the purpose of detecting continuous variation. One should note that our LM test is specifically designed to detect the temporal variation after the change point, while break tests can not tell what happens afterwards. As we said, martingale-type time variation is more harmful in the sense that the model can not be transformed into a stable one with constant factor loadings.

5 An Application

One application of factor models is nowcasting quarterly GDP growth using hundreds of predictors with different frequencies (Giannone et al. (2008) for Fed and Angelini et al. (2011) for ECB, for example). In practice, the performance is mixed and varies with the specific country to predict. It has good performance in the US and Euro Area, but fails to work well in the UK during and after the recession (Xu, 2014). One can use various refinements, such as variable pre-selection prior to factor extraction and downweighting past information, but the improvement is limited after the recent financial crisis. It implies that structural instability might be a major concern. One alternative solution is the factor model with time-varying loadings and stochastic volatility proposed in Del Negro & Otrok.
### Table 4: Power (stationary factors)

<table>
<thead>
<tr>
<th></th>
<th>$a = 1$</th>
<th>$a = 5$</th>
<th>$a = 10$</th>
<th>$a = 20$</th>
<th>$a = 30$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 50, T = 100$</td>
<td>0.448</td>
<td>0.688</td>
<td>0.778</td>
<td>0.831</td>
<td>0.850</td>
</tr>
<tr>
<td>$N = 50, T = 150$</td>
<td>0.562</td>
<td>0.805</td>
<td>0.871</td>
<td>0.906</td>
<td>0.915</td>
</tr>
<tr>
<td>$N = 50, T = 200$</td>
<td>0.620</td>
<td>0.858</td>
<td>0.914</td>
<td>0.939</td>
<td>0.942</td>
</tr>
<tr>
<td>$N = 100, T = 100$</td>
<td>0.518</td>
<td>0.769</td>
<td>0.827</td>
<td>0.871</td>
<td>0.893</td>
</tr>
<tr>
<td>$N = 100, T = 150$</td>
<td>0.640</td>
<td>0.860</td>
<td>0.915</td>
<td>0.943</td>
<td>0.959</td>
</tr>
<tr>
<td>$N = 100, T = 200$</td>
<td>0.720</td>
<td>0.908</td>
<td>0.946</td>
<td>0.971</td>
<td>0.974</td>
</tr>
<tr>
<td>$N = 150, T = 100$</td>
<td>0.544</td>
<td>0.781</td>
<td>0.842</td>
<td>0.880</td>
<td>0.903</td>
</tr>
<tr>
<td>$N = 150, T = 150$</td>
<td>0.662</td>
<td>0.874</td>
<td>0.924</td>
<td>0.948</td>
<td>0.962</td>
</tr>
<tr>
<td>$N = 150, T = 200$</td>
<td>0.742</td>
<td>0.917</td>
<td>0.953</td>
<td>0.975</td>
<td>0.980</td>
</tr>
<tr>
<td>$N = 200, T = 100$</td>
<td>0.565</td>
<td>0.789</td>
<td>0.852</td>
<td>0.894</td>
<td>0.904</td>
</tr>
<tr>
<td>$N = 200, T = 150$</td>
<td>0.682</td>
<td>0.869</td>
<td>0.927</td>
<td>0.953</td>
<td>0.962</td>
</tr>
<tr>
<td>$N = 200, T = 200$</td>
<td>0.736</td>
<td>0.924</td>
<td>0.958</td>
<td>0.974</td>
<td>0.982</td>
</tr>
</tbody>
</table>

### Table 5: Power (integrated factors)

<table>
<thead>
<tr>
<th></th>
<th>$a = 1$</th>
<th>$a = 5$</th>
<th>$a = 10$</th>
<th>$a = 20$</th>
<th>$a = 30$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 50, T = 100$</td>
<td>0.579</td>
<td>0.594</td>
<td>0.569</td>
<td>0.527</td>
<td>0.476</td>
</tr>
<tr>
<td>$N = 50, T = 150$</td>
<td>0.680</td>
<td>0.653</td>
<td>0.633</td>
<td>0.586</td>
<td>0.552</td>
</tr>
<tr>
<td>$N = 50, T = 200$</td>
<td>0.726</td>
<td>0.691</td>
<td>0.651</td>
<td>0.608</td>
<td>0.578</td>
</tr>
<tr>
<td>$N = 100, T = 100$</td>
<td>0.626</td>
<td>0.646</td>
<td>0.627</td>
<td>0.554</td>
<td>0.526</td>
</tr>
<tr>
<td>$N = 100, T = 150$</td>
<td>0.731</td>
<td>0.702</td>
<td>0.674</td>
<td>0.606</td>
<td>0.561</td>
</tr>
<tr>
<td>$N = 100, T = 200$</td>
<td>0.778</td>
<td>0.734</td>
<td>0.699</td>
<td>0.641</td>
<td>0.593</td>
</tr>
<tr>
<td>$N = 150, T = 100$</td>
<td>0.639</td>
<td>0.647</td>
<td>0.638</td>
<td>0.597</td>
<td>0.564</td>
</tr>
<tr>
<td>$N = 150, T = 150$</td>
<td>0.745</td>
<td>0.729</td>
<td>0.699</td>
<td>0.647</td>
<td>0.593</td>
</tr>
<tr>
<td>$N = 150, T = 200$</td>
<td>0.796</td>
<td>0.774</td>
<td>0.721</td>
<td>0.681</td>
<td>0.624</td>
</tr>
<tr>
<td>$N = 200, T = 100$</td>
<td>0.652</td>
<td>0.671</td>
<td>0.659</td>
<td>0.621</td>
<td>0.580</td>
</tr>
<tr>
<td>$N = 200, T = 150$</td>
<td>0.751</td>
<td>0.745</td>
<td>0.717</td>
<td>0.672</td>
<td>0.640</td>
</tr>
<tr>
<td>$N = 200, T = 200$</td>
<td>0.804</td>
<td>0.779</td>
<td>0.740</td>
<td>0.707</td>
<td>0.654</td>
</tr>
</tbody>
</table>
Figure 1: Power comparison for testing against martingale-type variation, $T = 100, N = 100, I(0)$ factors, block diagonal idiosyncratics. The horizontal axis gives the indicator of the strength of instability in factor loadings, while the vertical axis gives the power.
(2008). Therefore, it would be helpful to test whether there has been temporal instability since the crisis.

The raw dataset contains 134 UK series including survey and financial indicators, measures of demand, output, labor and housing activities, prices, and 18 US and Germany variables from 1990 to 2014. The series are either monthly or daily, and need to be transformed into a quarterly and stationary quantity. The details on the composition and transformation method of the dataset are listed in the Appendix. We use only the series for which there are less the 1/3 of missing data and adjust for outliers and missing data to create a balanced dataset.\(^2\) Although the tradition is to use as many series as possible, one reason behind some unsatisfactory performance is perhaps the adverse influence of uninformative predictors for the target variable to forecast. Boivin & Ng (2006) showed that as few as 40 pre-selected predictors can yield better forecasts than 147 predictors. Recently several attempts have been made to improve the forecasts based on estimated factors, including pre-selection of the predictors (Bai & Ng, 2008a) and boosting (Bai & Ng, 2009), among others. We pre-select 50 variables using the elastic net, which is known as an effective tool to perform variable selection and shrinkage simultaneously (Bai & Ng, 2008a).

We test whether there was temporal instability after January 2007. Onatski (2010)’s criterion indicates that the number of factors is 13, which is a little larger than that extracted from the US dataset. The relative rejection rate can be as high as 55\% from 2007:01 to 2009:12 shown in Figure 2, which implies the existence of instability during the financial crisis. It can be seen that the time variation was most serious around the middle of 2007 and gradually faded away afterwards.

\(^2\)Specifically, we replace the outliers and missing data with the median of a series.
Figure 2: Relative rejection rates from 2007:01 to 2009:12

6 Conclusions

In the paper, we adapt Nyblom (1989)’s statistic for high-dimensional factor models to test for the stable factor structure against considerable time variation in factor loadings in the form of martingales, which might affect the consistency of the factor estimator and therefore the forecasting performance of factor-augmented regressions. Under plausible conditions, the estimation error of factors and related variables is negligible for the asymptotic distribution of the test statistic. Although the tests designed for the alternatives of multiple breaks also have power for time-varying parameters, our statistic is more powerful to detect martingale-type time variation. We apply the test statistic to a UK dataset and find that the time variation was most serious around the middle of 2007 but gradually faded
away afterwards.

One might be interested in the joint null hypothesis that there is no time variation in the loadings of all variables. The pooled test statistic that combines $\hat{L}M_i$ is however not normally distributed, because the rate at which $\hat{L}M_i$ converges to $\tilde{L}M_i$ is not sufficiently high. We leave this for future research.

Appendix

In the appendix, we will prove the theorems stated in the paper. The proofs make extensive use of the Cauchy-Schwarz Inequality. For expositional simplicity, we only show the proof for the model with no cross-sectional dependence. Due to Assumption 4(g), it is straightforward to extend to the more general case.

A Proof of Theorem 1

Following Bai (2003),

$$F_t - H'F_t = \frac{1}{T} V^{-1}_{NT} \left( \sum_{s=1}^{T} \tilde{F}_s \gamma_{st} + \sum_{s=1}^{T} \tilde{F}_s \xi_{st} + \sum_{s=1}^{T} \tilde{F}_s \eta_{st} + \sum_{s=1}^{T} \tilde{F}_s \xi_{st} \right),$$

where $\gamma_{st} = E(u_s'u_t)/N$, $\xi_{st} = (u_s'u_t - E(u_s'u_t))/N$; $\eta_{st} = F_s' \Lambda_1' u_t / N$; $\xi_{st} = F_t' \Lambda_1' u_s / N$. For ease of notation, we set $T_a = 0$ but distinguish between $\tilde{T}$ and $T$ to generalize the results. $\delta_{NT} = \min(\sqrt{N}, \sqrt{T})$. We have the following lemmas under the null hypothesis and assumptions for all $N$, $T$, and $\tilde{T}$.

**Lemma 1.** $\sum_{t=1}^{\tilde{T}} || \sum_{s=1}^{T} (\tilde{F}_s - H'F_s)F_t' ||^2 = O_p(\frac{\tilde{T}^3}{\min(N^2, NT, T)})$.  

**Lemma 2.** $\sum_{t=1}^{\tilde{T}} || \sum_{s=1}^{T} (\tilde{F}_s - H'F_s)u_{it} ||^2 = O_p(\frac{\tilde{T}^3}{\min(N^2, NT, T)})$.  

**Lemma 3.** $\frac{1}{T} \sum_{t=1}^{\tilde{T}} (\tilde{F}_s - H'F_s)F_t' = O_p(\frac{1}{\min(N, \sqrt{NT}, T)})$.  

**Lemma 4.** $\frac{1}{T} \sum_{t=1}^{\tilde{T}} (\tilde{F}_s - H'F_s)u_{it} = O_p(\frac{1}{\min(N, \sqrt{NT}, T)})$.  

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Lemma 5. \( \frac{1}{T} \sum_{t=1}^{T} ||\hat{F}_t - H'F_t||^2 = O_p(\frac{1}{\min(N,T)}) \)

Lemma 6. \( \frac{1}{T} \sum_{t=1}^{T} (\hat{F}_t'F_t' - H'F_t(H'F_t)') = O_p(\frac{1}{\min(N,T^2)}). \) Therefore, \( \Sigma_F^{-1} = \Sigma_F^{-1} + o_p(1) \).

Proof of Lemma 1

By (1),

\[
\frac{1}{T} \sum_{t=1}^{T} ||\sum_{i=1}^{T} (\hat{F}_t - H'F_t) F_t'||^2 \leq 4||V_{N^T}||\left( \frac{1}{T} \sum_{t=1}^{T} ||\sum_{i=1}^{T} \hat{F}_t \gamma_{it} F_t'||^2 + \frac{1}{T} \sum_{t=1}^{T} ||\sum_{i=1}^{T} \hat{F}_t \xi_{it} F_t'||^2 \right) \\
+ \frac{1}{T} \sum_{t=1}^{T} ||\sum_{i=1}^{T} \tilde{F}_t \eta_{it} F_t'||^2 + \frac{1}{T} \sum_{t=1}^{T} ||\sum_{i=1}^{T} \tilde{F}_t \xi_{it} F_t'||^2 .
\]

First consider part I. By adding and subtracting terms,

\[
\frac{1}{T} \sum_{t=1}^{T} ||\sum_{i=1}^{T} \tilde{F}_t \gamma_{it} F_t'||^2 \leq 2(\frac{1}{T^2} \sum_{t=1}^{T} ||\sum_{i=1}^{T} \tilde{F}_t \gamma_{it} F_t'||^2 + \frac{1}{T} \sum_{t=1}^{T} ||\sum_{i=1}^{T} H'F_t \gamma_{it} F_t'||^2 ) .
\]

For the first term,

\[
\frac{1}{T^2} \sum_{t=1}^{T} ||\sum_{i=1}^{T} \tilde{F}_t \gamma_{it} F_t'||^2 \leq \frac{1}{T} \sum_{t=1}^{T} ||\sum_{i=1}^{T} \tilde{F}_t \gamma_{it} F_t'||^2 \sum_{t=1}^{T} ||\sum_{i=1}^{T} \tilde{F}_t \gamma_{it} E(u_t u_a)||^2 \\
\leq \frac{1}{T} \left( \frac{1}{T} \sum_{t=1}^{T} ||\tilde{F}_t - H'F_t||^2 \right) \left( \frac{1}{N} \sum_{i=1}^{N} E(u_a^2) \right)^2 \sum_{t=1}^{T} ||F_t'||^2 \\
= O_p(\frac{T^3}{T^2} \delta_{N^T}^2 ) ,
\]

by Theorem 1 in Bai & Ng (2002) and Assumptions 1 and 4(a).

Consider the second term,

\[
\frac{1}{T^2} \sum_{t=1}^{T} ||\sum_{i=1}^{T} H'F_t \gamma_{it} F_t'||^2 \leq \frac{1}{N^2T^2} \sum_{t=1}^{T} ||\sum_{i=1}^{T} \sum_{k=1}^{N} F_k E(u_k u_a)||^2 ||H'||^2 \\
\leq \frac{T}{T^2} ||\sum_{i=1}^{N} E(u_a^2)||^2 \sum_{t=1}^{T} ||F_t||^2 ||H'||^2 \\
= O_p(\frac{T^3}{T^2} ) .
\]
since \(||H|| = O_p(1)| from (Bai, 2003). Therefore \(I\) is \(O_p\left(\frac{T^3}{N^2} \delta_{NT}^{-2}\right)\).

Consider part II,

\[
\sum_{\tau=1}^{T} \left\| \frac{1}{T} \sum_{i=1}^{T} \sum_{j=1}^{T} (\hat{F}_i - H'F_i) \zeta_{ij} F_i' \right\|^2 \leq 2 \left( \frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{i=1}^{T} \sum_{j=1}^{T} (\hat{F}_i - H'F_i) \zeta_{ij} F_i' \right\|^2 + \frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{i=1}^{T} \sum_{j=1}^{T} H'F_i \zeta_{ij} F_i' \right\|^2 \right).
\]

By Assumption 4(c),

\[
\frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{i=1}^{T} \sum_{j=1}^{T} (\hat{F}_i - H'F_i) \zeta_{ij} F_i' \right\|^2 \leq \frac{1}{NT} \left( \frac{1}{T} \sum_{i=1}^{T} \left\| \hat{F}_i - H'F_i \right\|^2 \right) \sum_{i=1}^{T} \sum_{j=1}^{T} \left\| F_i \right\|^2 \sum_{i=1}^{T} \sum_{j=1}^{T} \left\| (u_{ij} - E(u_{ij})) \right\|^2 \leq \frac{T}{NT} \left( \frac{1}{T} \sum_{i=1}^{T} \left\| \hat{F}_i - H'F_i \right\|^2 \right) \sum_{i=1}^{T} \left\| F_i \right\|^2 \sum_{i=1}^{T} \sum_{j=1}^{T} \left\| (u_{ij} - E(u_{ij})) \right\|^2 = O_p\left(\frac{T^3}{N^2} \delta_{NT}^{-2}\right).
\]

By Assumption 4(d),

\[
\frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{i=1}^{T} \sum_{j=1}^{T} H'F_i \zeta_{ij} F_i' \right\|^2 \leq \frac{T}{NT} \sum_{i=1}^{T} \left\| \sum_{i=1}^{T} \sum_{j=1}^{T} F_i (u_{ij} - E(u_{ij})) \right\|^2 \sum_{i=1}^{T} \left\| F_i \right\|^2 \frac{1}{N} \sum_{i=1}^{T} \left\| (u_{ij} - E(u_{ij})) \right\|^2 = O_p\left(\frac{T^3}{N} \delta_{NT}^{-2}\right).
\]

Thus, II is \(O_p\left(\frac{T^3}{N} \delta_{NT}^{-2}\right)\).

Consider part III,

\[
\sum_{\tau=1}^{T} \left\| \frac{1}{T} \sum_{i=1}^{T} \sum_{j=1}^{T} F_i \eta_{ij} F_i' \right\|^2 \leq 2 \left( \frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{i=1}^{T} \sum_{j=1}^{T} (\hat{F}_i - H'F_i) \eta_{ij} F_i' \right\|^2 + \frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{i=1}^{T} \sum_{j=1}^{T} H'F_i \eta_{ij} F_i' \right\|^2 \right).
\]

For the first expression in the bracket,

\[
\frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{i=1}^{T} \sum_{j=1}^{T} (\hat{F}_i - H'F_i) \eta_{ij} F_i' \right\|^2 \leq \frac{T}{NT} \left( \frac{1}{T} \sum_{i=1}^{T} \left\| \hat{F}_i - H'F_i \right\|^2 \right) \sum_{i=1}^{T} \left\| F_i \right\|^2 \sum_{i=1}^{T} \sum_{j=1}^{T} \left\| \eta_{ij} \right\|^2 = O_p\left(\frac{T^2}{N^2} \delta_{NT}^{-2}\right).
\]

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For the second expression,

\[
\frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{t=1}^{T} \sum_{s=1}^{T} H' F_{s, \xi_{s} F'_{t}} \right\|^2 \leq \frac{T^2}{NT^2} \left( \sum_{s=1}^{T} \left\| F_{s} \right\|^2 \right) \frac{1}{T} \sum_{\tau=1}^{T} \left\| \sum_{t=1}^{T} \Lambda_{t} u_{t} F'_{t} \right\|^2 \left\| H' \right\|^2
\]

\[
= O_p\left( \frac{T^2}{N} \right).
\]

As a result, III is \( O_p\left( \frac{T^2}{N} \right) \).

Finally consider part IV.

\[
\sum_{\tau=1}^{T} \left\| \sum_{i=1}^{T} \sum_{s=1}^{T} (F_{s} - H' F_{s}) \xi_{s} F'_{t} \right\|^2 \leq 2 \left( \frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{t=1}^{T} \sum_{s=1}^{T} (F_{s} - H' F_{s}) \xi_{s} F'_{t} \right\|^2 + \frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{t=1}^{T} \sum_{s=1}^{T} H' F_{s, \xi_{s} F'_{t}} \right\|^2 \right).
\]

For the first term,

\[
\frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{i=1}^{T} \sum_{s=1}^{T} (F_{s} - H' F_{s}) \xi_{s} F'_{t} \right\|^2 \leq \frac{T}{NT} \left( \sum_{s=1}^{T} \left\| F_{s} - H' F_{s} \right\|^2 \right) \left( \sum_{t=1}^{T} \left\| \sum_{i=1}^{T} \Lambda_{t} u_{t} \right\|^2 \left( \sum_{i=1}^{T} \left\| F_{i} \right\|^2 \right) \right)
\]

\[
= \frac{T}{NT} O_p(\delta_{NT}^{-2}) O_p(T) O_p(T^2)
\]

\[
= O_p\left( \frac{T^3}{N} \delta_{NT}^{-2} \right).
\]

Consider the second term,

\[
\frac{1}{T^2} \sum_{\tau=1}^{T} \left\| \sum_{i=1}^{T} \sum_{s=1}^{T} H' F_{s, \xi_{s} F'_{t}} \right\|^2 \leq \frac{T}{NT} \left\| \sum_{s=1}^{T} F_{s, u_{s} \Lambda_{t}} \right\|^2 \left( \sum_{t=1}^{T} \left\| F_{t} \right\|^2 \right) \left\| H' \right\|^2
\]

\[
= O_p\left( \frac{T^3}{NT} \right).
\]

Thus, IV is \( O_p\left( \frac{T^3}{N} \delta_{NT}^{-2} \right) \). As \( ||V_{NT}|| = O_p(1) \) from Bai (2003), the proof is finished by combining these results.
Proof of Lemma 2

\[
\sum_{t=1}^{\hat{T}} \left\| \frac{1}{T} \sum_{i=1}^{T} (\hat{F}_i - H'F_s) u_i \right\|^2 \leq 4 \left\| V_{NT}^{-1} \right\| \left( \sum_{t=1}^{\hat{T}} \left\| \frac{1}{T} \sum_{i=1}^{T} \sum_{s=1}^{T} \hat{F}_s \eta_{is} u_i \right\|^2 \right) + \sum_{t=1}^{\hat{T}} \left\| \frac{1}{T} \sum_{i=1}^{T} \sum_{s=1}^{T} \hat{F}_s \xi_{is} u_i \right\|^2
\]

We begin with part I.

\[
\sum_{t=1}^{\hat{T}} \left\| \frac{1}{T} \sum_{i=1}^{T} \sum_{s=1}^{T} \hat{F}_s \eta_{is} u_i \right\|^2 \leq 2 \left( \frac{\hat{T}}{T^2} \sum_{t=1}^{\hat{T}} \left\| \sum_{i=1}^{T} (\hat{F}_s - H'F_s) \eta_{is} u_i \right\|^2 + \frac{1}{T^2} \sum_{t=1}^{\hat{T}} \left\| \sum_{i=1}^{T} H'F_s \eta_{is} u_i \right\|^2 \right).
\]

The first expression on the right hand side is bounded by

\[
\frac{T^2}{\hat{T}} \left( \frac{1}{T} \sum_{i=1}^{T} \left\| \hat{F}_s - H'F_s \right\|^2 \right) \left( \frac{1}{T} \sum_{i=1}^{T} \left\| \sum_{i=1}^{T} \eta_{is} u_i \right\|^2 \frac{1}{N} \sum_{j=1}^{N} E(u_j^2) \right),
\]

which is \(O_p(\frac{T^2}{\hat{T}} \delta_{NT}^{-2})\). The second term is bounded by

\[
\frac{T^2}{\hat{T}^2} \frac{1}{T} \sum_{i=1}^{\hat{T}} \left\| \frac{1}{\sqrt{T}} \sum_{i=1}^{T} F_s u_i \right\| \frac{1}{N} \sum_{j=1}^{N} E(u_j^2) \left\| H' \right\|^2 = O_p(\frac{T^2}{\hat{T}^2}).
\]

Thus, I is \(O_p(\frac{T^2}{\hat{T}} \delta_{NT}^{-2})\).

Part II is similar to the one in Lemma 1, which is \(O_p(\frac{T^2}{NT} \delta_{NT}^{-2})\).

Consider part III.

\[
\sum_{t=1}^{\hat{T}} \left\| \frac{1}{T} \sum_{i=1}^{T} \sum_{s=1}^{T} \hat{F}_s \eta_{is} u_i \right\|^2 \leq 2 \left( \frac{\hat{T}}{T^2} \sum_{t=1}^{\hat{T}} \left\| \sum_{i=1}^{T} (\hat{F}_s - H'F_s) \eta_{is} u_i \right\|^2 + \frac{1}{T^2} \sum_{t=1}^{\hat{T}} \left\| \sum_{i=1}^{T} H'F_s \eta_{is} u_i \right\|^2 \right).
\]

The first term is bounded by

\[
\frac{T}{NT} \left( \frac{\hat{T}}{T} \sum_{s=1}^{T} \left\| \hat{F}_s - H'F_s \right\|^2 \right) \sum_{s=1}^{T} \left\| F_s \right\|^2 \sum_{i=1}^{T} \left\| \eta_{is} u_i \right\|^2 \frac{1}{N} \sum_{j=1}^{N} \left\| \Lambda_j u_i \right\|^2,
\]

which is
$O_p\left(\frac{T^3}{N} \delta_{NT}^2\right)$. For the second term,

$$\frac{1}{T^2} \sum_{\tau_1=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} (\hat{F}_t F_s') \xi_{st} u_{t|s} \right|^2 \leq \frac{1}{NT^2} \sum_{s=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} \Lambda'_t u_t u_{t|s} \right|^2 ||H'||^2 \leq \frac{2T^2}{NT^2} \sum_{s=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} \Lambda'_t u_t u_{t|s} \right|^2 ||H'||^2 + \frac{2}{N\tilde{T}T^2} \sum_{s=1}^{\tilde{T}} \left( \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} \left| \Lambda'_t \right| \left| H'(u_{t|s}) \right| \right)^2 ||H'||^2 = O_p\left(\frac{T^2}{N}\right) + O_p\left(\frac{T^3}{N^2}\right),$$

by Assumption 2, 4(b) and 4(h). III is thus $O_p\left(\frac{T^3}{N\min(N,T)}\right)$.

Consider part IV.

$$\sum_{\tau_1=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} (\hat{F}_t - H'F_s) \xi_{st} u_{t|s} \right|^2 \leq 2 \left( \frac{1}{T^2} \sum_{s=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} (\hat{F}_t - H'F_s) \xi_{st} u_{t|s} \right|^2 + \frac{1}{T^2} \sum_{t=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} H'F_s \xi_{st} u_{t|s} \right|^2 \right).$$

For the first expression in the bracket,

$$\frac{1}{T^2} \sum_{t=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} (\hat{F}_t - H'F_s) \xi_{st} u_{t|s} \right|^2 \leq \frac{T^2}{N} \left( \frac{1}{T} \sum_{s=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} \Lambda'_t u_t u_{t|s} \right|^2 \right) \frac{1}{T} \sum_{t=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} F_t u_{t|s} \right|^2 = O_p\left(\frac{T^2}{N} \delta_{NT}^2\right).$$

The second term is bounded by $\frac{T^2}{N} \sum_{s=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{\tau} \Lambda'_t u_t u_{t|s} \right|^2 ||H'||^2 = O_p\left(\frac{T^3}{NT^2}\right)$.

Thus, IV is $O_p\left(\frac{T^3}{N} \delta_{N\tilde{T}}^2\right)$.

As a result, $\sum_{\tau_1=1}^{\tilde{T}} \left| \sum_{t=1}^{\tilde{T}} (\hat{F}_t - H'F_s) u_{t|s} \right|^2 = 4 ||V_{NT}^{-1} || (I + II + III + IV) = O_p\left(\frac{T^3}{\min(N^2, NT^2)}\right)$.

**Proof of Lemma 3**

The proof of Lemma 3 is straightforwardly adapted from that of Lemma B.2 of Bai (2003). The difference between Lemma 3 and Lemma B.2 is that Lemma 3 is a mean over $\tilde{T}$ periods while Lemma B.2 is a mean over $T$ periods. The factors are estimated using the data of $T$ periods in both cases, however. As a result, they can be of different orders in probability. Specifically,
\[
\frac{1}{T} \sum_{i=1}^{T} (\hat{F}_i - H'F_i)F_i' = V_{TF}^{-1} \frac{1}{T} \sum_{i=1}^{T} \sum_{x=1}^{T} \hat{F}_x \gamma_{dx} F_i' + V_{TF}^{-1} \frac{1}{T} \sum_{i=1}^{T} \sum_{x=1}^{T} \hat{F}_x \xi_{dx} F_i' \\
+ V_{TF}^{-1} \frac{1}{T} \sum_{i=1}^{T} \sum_{x=1}^{T} F_i \eta_{dx} F_i' + V_{TF}^{-1} \frac{1}{T} \sum_{i=1}^{T} \sum_{x=1}^{T} F_i \xi_{dx} F_i'.
\]

First consider I.

\[
\frac{1}{TT} \sum_{i=1}^{T} \sum_{x=1}^{T} \hat{F}_x \gamma_{dx} F_i' = \frac{1}{TT} \sum_{i=1}^{T} \sum_{x=1}^{T} (\hat{F}_x - H'F_x) \gamma_{dx} F_i' + \frac{1}{TT} \sum_{i=1}^{T} \sum_{x=1}^{T} H'F_x \gamma_{dx} F_i',
\]

where

\[
|| \frac{1}{TT} \sum_{i=1}^{T} \sum_{x=1}^{T} (\hat{F}_x - H'F_x) \gamma_{dx} F_i' || \leq \frac{1}{TT} (\sum_{x=1}^{T} ||(\hat{F}_x - H'F_x)||^2)^{1/2} (\sum_{x=1}^{T} \sum_{i=1}^{T} \frac{1}{N} E(u'_x u_x)^2)^{1/2} \\
\leq \frac{1}{TT} (\sum_{x=1}^{T} ||(\hat{F}_x - H'F_x)||^2)^{1/2} (\sum_{x=1}^{T} \sum_{i=1}^{T} \frac{1}{N} E(u'_x u_x)^2)^{1/2} (\sum_{i=1}^{T} ||F_i'||^2)^{1/2} \\
= O_p\left(\frac{1}{\sqrt{T \min(\sqrt{N}, \sqrt{T})}}\right),
\]

and

\[
|| \frac{1}{TT} \sum_{i=1}^{T} \sum_{x=1}^{T} H'F_x \gamma_{dx} F_i' || = || \frac{1}{TT} \sum_{i=1}^{T} H'F_i' \gamma_{i} || \\
\leq \frac{||H'||}{TT} (\sum_{i=1}^{T} ||F_i'||^2)^{1/2} (\sum_{i=1}^{T} ||\gamma_{i}||^2)^{1/2} \\
= O_p\left(\frac{1}{T}\right).
\]

Thus, I is $O_p\left(\frac{1}{\sqrt{T \min(\sqrt{N}, \sqrt{T})}}\right)$.

Consider II.

\[
\frac{1}{TT} \sum_{i=1}^{T} \sum_{x=1}^{T} \hat{F}_x \xi_{dx} F_i' = \frac{1}{TT} \sum_{i=1}^{T} \sum_{x=1}^{T} (\hat{F}_x - H'F_x) \xi_{dx} F_i' + \frac{1}{TT} \sum_{i=1}^{T} \sum_{x=1}^{T} H'F_x \xi_{dx} F_i',
\]

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where

\[
\frac{1}{TT} \sum_{t=1}^{\bar{T}} \sum_{s=1}^{T} (\hat{F}_s - H'F_s) \zeta_{st} F'_i \leq \frac{1}{TT} \left( \sum_{s=1}^{T} \| (\hat{F}_s - H'F_s) \|^2 \right)^{1/2} \left( \sum_{s=1}^{T} \frac{1}{N} \| u'_s u_t - E(u'_s u_t) \|^2 \right)^{1/2} \\
\leq \frac{1}{TT} \left( \sum_{s=1}^{T} \| (\hat{F}_s - H'F_s) \|^2 \right)^{1/2} \left( \sum_{s=1}^{T} \frac{1}{N} \| u'_s u_t - E(u'_s u_t) \|^2 \right)^{1/2} \left( \sum_{t=1}^T \| F'_i \|^2 \right)^{1/2} \\
= O_p\left( \frac{1}{\sqrt{N} \min(\sqrt{N}, \sqrt{T})} \right),
\]

by Assumption 4(c), and

\[
\frac{1}{TT} \sum_{t=1}^{\bar{T}} \sum_{s=1}^{T} H'F_s \zeta_{st} F'_i \leq \frac{1}{TT} \left( \sum_{t=1}^{\bar{T}} \left( \sum_{s=1}^{T} F_s \frac{1}{N} \| u'_s u_t - E(u'_s u_t) \|^2 \right) \right)^{1/2} \left( \sum_{t=1}^{\bar{T}} \| F'_i \|^2 \right)^{1/2} \\
= O_p\left( \frac{1}{\sqrt{NT}} \right),
\]

by Assumption 4(d). As a result, \( T \) is \( O_p\left( \frac{1}{\sqrt{N} \min(\sqrt{N}, \sqrt{T})} \right) \).

Then consider \( III \).

\[
\frac{1}{TT} \sum_{t=1}^{\bar{T}} \sum_{s=1}^{T} \hat{F}_s \eta_{st} F'_i = \frac{1}{TT} \sum_{t=1}^{\bar{T}} \sum_{s=1}^{T} (\hat{F}_s - H'F_s) \eta_{st} F'_i + \frac{1}{TT} \sum_{t=1}^{\bar{T}} \sum_{s=1}^{T} H'F_s \eta_{st} F'_i.
\]

The first term is

\[
\frac{1}{TT} \sum_{t=1}^{\bar{T}} \sum_{s=1}^{T} (\hat{F}_s - H'F_s) \eta_{st} F'_i \leq \frac{1}{TT} \left( \sum_{s=1}^{T} \| (\hat{F}_s - H'F_s) \|^2 \right)^{1/2} \left( \sum_{s=1}^{T} \frac{1}{N} \| \Lambda'_s u'_s F'_i \|^2 \right)^{1/2} \\
\leq \frac{1}{TT} \left( \sum_{s=1}^{T} \| (\hat{F}_s - H'F_s) \|^2 \right)^{1/2} \left( \sum_{s=1}^{T} \frac{1}{N} \| \Lambda'_s u'_s F'_i \|^2 \right)^{1/2} \left( \sum_{s=1}^{T} \| F'_i \|^2 \right)^{1/2} \\
= O_p\left( \frac{1}{\sqrt{NT} \min(\sqrt{N}, \sqrt{T})} \right).
\]

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The second term is

\[
\| \frac{1}{TT} \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{T} H' F_s \eta_{st} F'_s \| = \| \frac{1}{TT} \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{T} H' F_s \frac{1}{N} F'_s \Lambda'_{1} u_s F'_s \| = \| \frac{1}{NTT} \| \sum_{s=1}^{T} F_s F'_s \| \| \sum_{t=1}^{\tilde{T}} \Lambda'_{1} u_s F'_s \| = O_p(\frac{1}{\sqrt{NT}}).
\]

Thus III is therefore \(O_p(\frac{1}{\sqrt{NT}}).\)

Finally consider IV.

\[
\frac{1}{TT} \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{T} \hat{F}_s \xi_{st} F'_s = \frac{1}{TT} \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{T} (\hat{F}_s - H' F_s) \xi_{st} F'_s + \frac{1}{TT} \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{T} H' F_s \xi_{st} F'_s.
\]

The first term is

\[
\| \frac{1}{TT} \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{T} (\hat{F}_s - H' F_s) \xi_{st} F'_s \| \leq \frac{1}{TT} \left( \sum_{s=1}^{T} \| (\hat{F}_s - H' F_s) \| \| \sum_{t=1}^{\tilde{T}} \frac{1}{N} F'_s \Lambda'_{1} u_s F'_s \| \right)^{\frac{1}{2}} \leq \frac{1}{TT} \left( \sum_{s=1}^{T} \| (\hat{F}_s - H' F_s) \| \| \sum_{s=1}^{T} \frac{1}{N} F'_s \Lambda'_{1} u_s \| \| \sum_{t=1}^{\tilde{T}} F'_s \| \right)^{\frac{1}{2}} = O_p(\frac{1}{\sqrt{N \min(\sqrt{N}, \sqrt{T})}}).
\]

The second term is

\[
\| \frac{1}{TT} \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{T} H' F_s \xi_{st} F'_s \| = \| \frac{1}{TT} \sum_{t=1}^{\tilde{T}} \sum_{s=1}^{T} H' F_s \frac{1}{N} F'_s \Lambda'_{1} u_s F'_s \| = \| \frac{1}{NTT} \| \sum_{s=1}^{T} F_s \xi_{st} F'_s \| \| \sum_{t=1}^{\tilde{T}} F'_s \| = O_p(\frac{1}{\sqrt{NT}}).
\]

Thus IV is \(O_p(\frac{1}{\sqrt{N \min(\sqrt{N}, \sqrt{T})}}).\) As a result, \(I + II + III + IV = O_p(\frac{1}{\sqrt{T \min(\sqrt{N}, \sqrt{T})}}) + O_p(\frac{1}{\sqrt{N \min(\sqrt{N}, \sqrt{T})}}) + O_p(\frac{1}{\sqrt{NT}}) = O_p(\frac{1}{\min(N, \sqrt{NT}, T)}).

Proofs of Lemmas 4 and 5
The proofs of Lemmas 4 and 5 are straightforwardly adapted from those of Lemma B.1 of Bai (2003) and Theorem 1 of Bai & Ng (2002), respectively. The difference is still that the average is taken over \( \hat{T} \) periods in my case, but is taken over \( T \) periods in Bai & Ng (2002) and Bai (2003). The adaptation is therefore very similar to that in the proof of Lemma 3, so we omit it here.

**Proof of Lemma 6**

\[
\frac{1}{\hat{T}} \sum_{t=1}^{\hat{T}} (\hat{F}_t \hat{F}_t' - H'F_t (H'F_t)' = \frac{1}{\hat{T}} \sum_{t=1}^{T} (\hat{F}_t - H'F_t)(\hat{F}_t' - (H'F_t)' + \frac{1}{\hat{T}} \sum_{t=1}^{T} (\hat{F}_t - H'F_t)(H'F_t)'
\]

\[
+ \frac{1}{\hat{T}} \sum_{t=1}^{T} H'F_t (\hat{F}_t' - (H'F_t)')
\]

\[
= O_p \left( \frac{1}{\min(N, \sqrt{\hat{T}T})} \right),
\]

by Lemmas 3 and 5.

**Proof of Theorem 1**

Let \( \hat{d}_{it} = \Sigma_{p}^{-2} \hat{s}_{it} \) where \( \Sigma_{p}^{-2} = (\Sigma_{p}^{-2})' \Sigma_{p}^{-2} \) by Cholesky decomposition. Similarly, \( \hat{d}_{it} = \Sigma_{p}^{-2} \hat{s}_{it} \). As \( \hat{LM}_l - \hat{LM}_i = \frac{1}{\hat{T}^2 \hat{V}_{it}} \sum_{t=1}^{\hat{T}} \hat{d}_{it} \hat{d}_{it} - \frac{1}{\hat{T}^2 \hat{V}_{it}} \sum_{t=1}^{\hat{T}} \hat{d}_{it} \hat{d}_{it} \) and \( \hat{V}_{it}^{-1} \hat{V}_{it}^{-1} \) by Assumption 4(giii), we need to study the order of \( \frac{1}{\hat{T}^2} \sum_{t=1}^{\hat{T}} \hat{d}_{it} \hat{d}_{it} - \frac{1}{\hat{T}^2} \sum_{t=1}^{\hat{T}} \hat{d}_{it} \hat{d}_{it} \). We have

\[
\frac{1}{\hat{T}^2} \sum_{t=1}^{\hat{T}} \hat{d}_{it} \hat{d}_{it} - \frac{1}{\hat{T}^2} \sum_{t=1}^{\hat{T}} \hat{d}_{it} \hat{d}_{it} = \frac{1}{\hat{T}^2} \sum_{t=1}^{\hat{T}} (\hat{d}_{it} - \hat{d}_{it})' (\hat{d}_{it} + \hat{d}_{it})
\]

\[
= \frac{1}{\hat{T}^2} \sum_{t=1}^{\hat{T}} (\hat{d}_{it} - \hat{d}_{it})' (\hat{d}_{it} - \hat{d}_{it}) + \frac{2}{\hat{T}^2} \sum_{t=1}^{\hat{T}} (\hat{d}_{it} - \hat{d}_{it})' \hat{d}_{it}
\]

\[
\leq \frac{1}{\hat{T}^2} \sum_{t=1}^{\hat{T}} ||\hat{d}_{it} - \hat{d}_{it}||^2 + 2 \frac{1}{\hat{T}^2} \sum_{t=1}^{\hat{T}} ||\hat{d}_{it} - \hat{d}_{it}||^2 \hat{d}_{it} \hat{d}_{it} (\frac{1}{\hat{T}^2} \sum_{t=1}^{\hat{T}} \hat{d}_{it} \hat{d}_{it})^2 = O_p(1)
\]
where
\[
\frac{1}{T^2} \sum_{t=1}^{T} ||d_{it} - \hat{d}_{it}||^2 = \frac{1}{T^2} \sum_{t=1}^{T} ||\hat{\Sigma}_{it}^{-\frac{1}{2}} \hat{s}_{it} - \Sigma_{it}^{-\frac{1}{2}} s_{it}||^2
\]
\[
\leq 2 ||\hat{\Sigma}_{it}^{-\frac{1}{2}}||^2 \frac{1}{T^2} \sum_{t=1}^{T} ||(\hat{s}_{it} - s_{it})||^2 + 2 ||\hat{\Sigma}_{it}^{-\frac{1}{2}} - \Sigma_{it}^{-\frac{1}{2}}||^2 \frac{1}{T^2} \sum_{t=1}^{T} ||s_{it}||^2.
\]

Thus, the key is the order of \(\frac{1}{T^2} \sum_{t=1}^{T} ||(\hat{s}_{it} - s_{it})||^2\). For expositional simplicity, we only show the remaining proof for the models with no cross-sectional dependence. Due to Assumption 4(g), it is straightforward to extend to the more general case. In the models with cross-sectional independence, \(\hat{V}_{i}^{-1} X_t = \hat{V}_{ii}^{-1} X_t\) and \(V_i^{-1} X_t = V_{ii}^{-1} X_t\). Therefore,

\[
\hat{\Sigma}_{it} - \Sigma_{it} = \left( \sum_{t=1}^{T} (\hat{F}_i V_{ii}^{-1} X_t) - \sum_{t=1}^{T} F_i (\sum_{m=1}^{M} \hat{F}_m F_m')^{-1} \sum_{m=1}^{M} F_m V_{im}^{-1} X_{im} \right) - \left( \sum_{t=1}^{T} (\hat{F}_i V_{ii}^{-1} X_t) - \sum_{t=1}^{T} F_i (\sum_{m=1}^{M} \hat{F}_m F_m')^{-1} \sum_{m=1}^{M} F_m V_{ii}^{-1} X_{im} \right)
\]
\[
= \sum_{t=1}^{T} (\hat{F}_i - F_i)(F_i - F_i)'(\sum_{m=1}^{M} \hat{F}_m F_m')^{-1} \sum_{m=1}^{M} F_m V_{ii}^{-1} X_{im}.
\]

Therefore,
\[
\frac{1}{T^2} \sum_{t=1}^{T} ||(\hat{s}_{it} - s_{it})||^2 \leq 2 \frac{1}{T^2} \sum_{t=1}^{T} ||l||^2 + 3 \frac{1}{T^2} \sum_{t=1}^{T} ||l||^2 + 3 \frac{1}{T^2} \sum_{t=1}^{T} ||III||^2.
\]

First consider \(\sum_{t=1}^{T} ||l||^2\). Since \(\hat{V}_{ii}^{-1} P_{it} V_{ii}^{-1}\), we only need to compute the order of \(I = \sum_{t=1}^{T} (\hat{F}_i - F_i) X_{it}\). Note that
\[
\sum_{t=1}^{T} ||l||^2 \leq 2 \sum_{t=1}^{T} ||\sum_{i=1}^{I} (\hat{F}_i - F_i) F_i' \lambda_{i1}||^2 + 2 \sum_{t=1}^{T} ||\sum_{i=1}^{I} (\hat{F}_i - F_i) u_{it}||^2.
\]

By Lemmas 1 and 2, \(\sum_{t=1}^{T} ||l||^2 = O_p(\min(N^2, NT, T^2)) + O_p(\min(N^2, NT, T^2)) = O_p(\min(N^2, NT, T^2))\).

Consider \(\sum_{t=1}^{T} ||l||^2\). Because of \(\hat{V}_{ii}^{-1} P_{it} V_{ii}^{-1}\), the key point is the order of \((\sum_{m=1}^{M} \hat{F}_m F_m')^{-1} \sum_{m=1}^{M} F_m X_{im} - \sum_{m=1}^{M} F_m X_{im})\).
By Lemma 1 and 5, Consequently, Lemma 7.

\[
(\sum_{m=1}^{T} \tilde{F}_m \tilde{F}_m')^{-1} \sum_{m=1}^{T} \tilde{F}_m X_{im}. \quad \text{By Lemmas 3, 4 and 6,}
\]
\[
(\sum_{m=1}^{T} \tilde{F}_m \tilde{F}_m')^{-1} \sum_{m=1}^{T} \tilde{F}_m X_{im} - (\sum_{m=1}^{T} \tilde{F}_m \tilde{F}_m')^{-1} \sum_{m=1}^{T} \tilde{F}_m X_{im} = O_p\left(\frac{1}{\min(N, \sqrt{NT}, \bar{T})}\right).
\]

Consequently, \(\sum_{\tau=1}^{T} |I|^2 = O_p\left(\frac{3}{\min(N^2, NT, \bar{T}^2)}\right)\).

Finally consider \(\sum_{\tau=1}^{T} ||II||^2\). As
\[
\sum_{\tau=1}^{T} \sum_{t=1}^{T} ||(\tilde{F}_t - \hat{F}_t)(\hat{F}_t' - \hat{F}_t')||^2 \\
\leq \sum_{\tau=1}^{T} \sum_{t=1}^{T} ||(\tilde{F}_t - \hat{F}_t)(\hat{F}_t' - \hat{F}_t')||^2 + \sum_{t=1}^{T} \sum_{t=1}^{T} ||(\tilde{F}_t - \hat{F}_t)(\hat{F}_t' - \hat{F}_t')||^2 \\
\leq 3 \sum_{\tau=1}^{T} \sum_{t=1}^{T} ||(\tilde{F}_t - \hat{F}_t)(\hat{F}_t' - \hat{F}_t')||^2 + 6 \sum_{t=1}^{T} \sum_{t=1}^{T} ||(\tilde{F}_t - \hat{F}_t)(\hat{F}_t' - \hat{F}_t')||^2 \\
\leq 3T(\sum_{\tau=1}^{T} ||F_t - \hat{F}_t||^2)^2 + 6 \sum_{t=1}^{T} \sum_{t=1}^{T} ||(\tilde{F}_t - \hat{F}_t)(\hat{F}_t' - \hat{F}_t')||^2.
\]

By Lemma 1 and 5, \(\sum_{\tau=1}^{T} ||III||^2 = O_p\left(\frac{3}{\min(N^2, NT, \bar{T}^2)}\right)\). As a result, \(\frac{1}{\bar{T}^2} \sum_{\tau=1}^{T} ||(\tilde{s}_{\tau t} - \hat{s}_{\tau t})||^2 = O_p\left(\frac{T}{\sqrt{\min(N^2, NT, \bar{T}^2)}}\right)\) which leads to \(\hat{L}_{M_t} - \bar{L}_{M_t} = O_p\left(\frac{\sqrt{T}}{\sqrt{\min(N^2, NT, \bar{T}^2)}}\right)\).

### B Proof of Theorem 2

Define \(\tilde{e}_{ut} = \tilde{T} e_{ut}\) such that \(E(\tilde{e}_{ut} \tilde{e}_{ut}') = a\Sigma_{\tilde{e}}\). Under the local alternative,
\[
\tilde{F}_t - H'F_t = \frac{1}{T} V_{NT}^{-1} \left( \sum_{s=1}^{T} \tilde{F}_s \gamma_{st} + \sum_{s=1}^{T} \tilde{F}_s \zeta_{st} + \sum_{s=1}^{T} \tilde{F}_s \eta_{st} + \sum_{s=1}^{T} \tilde{F}_s \xi_{st} + \frac{1}{N} \left( \sum_{s=1}^{T} \tilde{F}_s \delta_{st} \lambda_1' v_t + \sum_{s=1}^{T} \tilde{F}_s \delta_{st} \lambda_1' v_t + \sum_{s=1}^{T} \tilde{F}_s \delta_{st} u' s v_t + \sum_{s=1}^{T} \tilde{F}_s \delta_{st} u' s v_t \right) \right).
\]

**Lemma 7.** \(\sum_{\tau=1}^{T} \sum_{t=1}^{T} (\tilde{F}_t - \hat{F}_t) v_{ut} ||^2 = O_p\left(\frac{T^2}{\min(N, \sqrt{NT}, \bar{T})}\right)\).

**Lemma 8.** \(\frac{1}{T^2} \sum_{t=1}^{T} (\tilde{F}_t - H'F_t) v_{ut} = O_p\left(\frac{1}{\sqrt{T \min(N, \sqrt{NT}, \bar{T})}}\right)\).

**Proof of Lemma 7**

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By (6),

$$\sum_{t=1}^{T} \left| \sum_{i=1}^{T} (\hat{F}_i - H'F_i)v_{it} \right|^2 \leq \frac{9}{N^2} \sum_{t=1}^{T} \left| \sum_{i=1}^{T} \sum_{s=1}^{T} \hat{F}_s \gamma_{st} v_{it} \right|^2 + \sum_{t=1}^{T} \left| \sum_{i=1}^{T} \sum_{s=1}^{T} \hat{F}_s \xi_{st} v_{it} \right|^2$$

$$+ \frac{1}{T} \sum_{i=1}^{T} \left| \sum_{s=1}^{T} \sum_{m=1}^{M} \tilde{e}_{im} \right|^2 + \frac{1}{T} \sum_{i=1}^{T} \left| \sum_{s=1}^{T} \sum_{m=1}^{M} \tilde{e}_{im} \right|^2$$

$$+ \frac{1}{NT} \sum_{i=1}^{T} \sum_{s=1}^{T} \sum_{m=1}^{M} \tilde{e}_{im} \sum_{i=1}^{T} \sum_{s=1}^{T} \sum_{m=1}^{M} \tilde{e}_{im} E(u_i u_{it})$$

We begin with part I.

$$\sum_{t=1}^{T} \left| \sum_{i=1}^{T} \sum_{s=1}^{T} \hat{F}_s \gamma_{st} v_{it} \right|^2 \leq \frac{1}{T^2} \sum_{i=1}^{T} \left| \sum_{s=1}^{T} \sum_{m=1}^{M} \tilde{e}_{im} \right|^2 + \frac{1}{T^2} \sum_{i=1}^{T} \left| \sum_{s=1}^{T} \sum_{m=1}^{M} \tilde{e}_{im} \right|^2$$

Consider the first term,

$$\frac{1}{T^2} \sum_{i=1}^{T} \left| \sum_{s=1}^{T} \sum_{m=1}^{M} (\hat{F}_s - H'F_i) \gamma_{st} F'_i \tilde{e}_{im} \right|^2 \leq \frac{1}{T^2} \sum_{i=1}^{T} \left| \sum_{s=1}^{T} \sum_{m=1}^{M} (\hat{F}_s - H'F_i) \right|^2$$

$$\frac{1}{T^2} \sum_{i=1}^{T} \left| \sum_{s=1}^{T} \sum_{m=1}^{M} \tilde{e}_{im} \right|^2 \leq \frac{1}{T^2} \left( \frac{1}{T} \sum_{i=1}^{T} \left| \hat{F}_s - H'F_i \right|^2 \right) \frac{1}{T} \sum_{i=1}^{T} \left| \sum_{m=1}^{M} \tilde{e}_{im} \right|^2$$

$$= O_p(\frac{T^2}{T \delta_{NT}^2})$$
as \((\sum_{t=1}^{\tilde{T}} ||F'_t \sum_{m=1} T \bar{e}_{im}||)^2 \leq \sum_{t=1}^{\tilde{T}} ||F'_t||^2 \sum_{m=1} T \bar{e}_{im}||^2 = O_p(\tilde{T}^3)\).

For the second term,

\[
\frac{1}{T^2\tilde{T}^2} \sum_{t=1}^{\tilde{T}} ||\sum_{t=1}^{T} \sum_{i=1}^T F'_t \gamma_i \eta_i F'_i \sum_{m=1} T \bar{e}_{im}||^2 \leq \frac{1}{T^2\tilde{T}^2} \sum_{t=1}^{\tilde{T}} \sum_{i=1}^T (F'_t - H'F_i) \eta_i F'_i \sum_{m=1} T \bar{e}_{im}||^2 \leq O_p(\tilde{T}^2),
\]

because \(\sum_{t=1}^{\tilde{T}} ||F'_t \sum_{m=1} T \bar{e}_{im}||^2 \leq \tilde{T} \sum_{t=1}^{\tilde{T}} ||F'_t||^4 \sum_{m=1} T \bar{e}_{im}||^2 = O_p(\tilde{T}^4)\).

It follows that \(I = O_p(\tilde{T}^2 \delta_{NT}^{-2})\).

Part II is bounded by \(O_p(\tilde{T}^2 \delta_{NT}^{-2})\) proved in a similar manner. Consider part III.

\[
\sum_{t=1}^{\tilde{T}} ||\sum_{t=1}^{T} \sum_{i=1}^T F'_i \gamma_i \eta_i F'_i \sum_{m=1} T \bar{e}_{im}||^2 \leq \sum_{t=1}^{\tilde{T}} ||\sum_{t=1}^{T} (F'_t - H'F_i) \eta_i F'_i \sum_{m=1} T \bar{e}_{im}||^2 \leq O_p(\tilde{T}^4),
\]

As \(\sum_{t=1}^{\tilde{T}} ||\sum_{t=1}^{T} \sum_{i=1}^T \Lambda'_i \eta_i F'_i \sum_{m=1} T \bar{e}_{im}||^2 \leq \sum_{t=1}^{\tilde{T}} ||\sum_{t=1}^{T} \Lambda'_i \eta_i F'_i \sum_{m=1} T \bar{e}_{im}||^2 = O_p(\tilde{T}^4)\), the first term is

\[
\frac{1}{T^2\tilde{T}^2} \sum_{t=1}^{\tilde{T}} ||\sum_{t=1}^{T} (F'_t - H'F_i) \eta_i F'_i \sum_{m=1} T \bar{e}_{im}||^2 \leq \frac{1}{T^2\tilde{T}^2} \sum_{t=1}^{\tilde{T}} ||F'_t - H'F_i||^2 \leq O_p(\tilde{T}^2 \delta_{NT}^{-2}).
\]

Similarly, the second term is \(O_p(\tilde{T}^2 \delta_{NT}^{-2})\). As a result, \(III = O_p(\tilde{T}^2 \delta_{NT}^{-2})\). Part IV is similarly \(O_p(\tilde{T}^2 \delta_{NT}^{-2})\).

Now study part V.

\[
\sum_{t=1}^{\tilde{T}} ||\sum_{t=1}^{T} \sum_{i=1}^T F'_i \Lambda'_i \eta_i F'_i \sum_{m=1} T \bar{e}_{im}||^2 \leq \sum_{t=1}^{\tilde{T}} ||\sum_{t=1}^{T} (F'_t - H'F_i) \Lambda'_i \eta_i F'_i \sum_{m=1} T \bar{e}_{im}||^2 + \sum_{t=1}^{\tilde{T}} ||\sum_{t=1}^{T} (H'F_i \Lambda'_i F'_i \sum_{m=1} T \bar{e}_{im}||^2 ||^2).\]

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The first term is bounded by
\[
\frac{1}{N^2T^3} \left( \frac{T}{T} \sum_{s=1}^{T} \left( \sum_{t=1}^{\tau} |F_t - H^t F_t|^2 \right) \right) \sum_{s=1}^{T} \left( \sum_{t=1}^{\tau} (F_t')^2 \right) \sum_{t=1}^{\tau} (\sum_{m=1}^{\ell} e_{im})^2 = O_p \left( \frac{T}{\sqrt{N^2 T^2}} \right),
\]
because \( \sum_{t=1}^{T} (\sum_{m=1}^{\ell} e_{im})^2 \leq T (\sum_{t=1}^{T} |F_t|^2 |(\sum_{m=1}^{\ell} e_{im})^2 |^2 \leq T^2 \sum_{t=1}^{T} |F_t|^4 \sum_{m=1}^{\ell} |e_{im}|^4 = O_p(T^3).\)

For the second term,
\[
\frac{1}{N^2 T^2 T^4} \sum_{t=1}^{T} \sum_{i=1}^{\tau} \sum_{j=1}^{\tau} H^t F_t F_t' \sum_{m=1}^{\ell} e_{im} F_t' F_t' \sum_{m=1}^{\ell} e_{im} |^2 \leq \frac{1}{N^2 T^2 T^4} \sum_{t=1}^{T} \sum_{i=1}^{\tau} \sum_{j=1}^{\tau} (F_t')^2 \sum_{m=1}^{\ell} \sum_{m=1}^{\ell} (e_{im})^2 |^2 |H_t'|^2
\]
\[
= O_p \left( \frac{T}{N^2} \right).
\]

Thus, \( V = O_p \left( \frac{T}{N^2} \right) = o_p \left( \frac{T^2}{\min(N, T^2)} \right). \) Part VI – IX are all \( o_p \left( \frac{T^2}{\min(N, T^2)} \right) \) in a similar manner. The lemma is finally proved by combining the results of these parts.

**Proof of Lemma 8**

The proof of Lemma 8 is similar to that of Lemma 7 and is omitted here.

**Proof of Theorem 2**

In a similar manner, it can be shown that Lemmas 1-6 still hold under the local alternative \( h_{NT} = a/T^2 \) with \( a \) fixed (The proofs are available on request). The proof is similar to that of Theorem 1. (4) still works.

Now (5) changes to
\[
\sum_{t=1}^{\tilde{T}} \sum_{i=1}^{\tau} |l|^2 \leq 3 \sum_{t=1}^{\tilde{T}} \sum_{i=1}^{\tau} |(\tilde{F}_t - \tilde{F}_t) F_t' \lambda_{it}|^2 + 3 \sum_{t=1}^{\tilde{T}} \sum_{i=1}^{\tau} |(\tilde{F}_t - \tilde{F}_t) v_{it}|^2 + 3 \sum_{t=1}^{\tilde{T}} \sum_{i=1}^{\tau} |(\tilde{F}_t - \tilde{F}_t) u_{it}|^2.
\]

By Lemma 7, \( \sum_{t=1}^{\tilde{T}} \sum_{i=1}^{\tau} |l|^2 \) is same as before. By Lemma 8, \( \sum_{t=1}^{\tilde{T}} |l|^2 \) is same as before. The order of III is also unchanged. It follows that Theorem 2 is proved.

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C Proof of Theorem 3

The proof is similar to that for Theorem 1, so we only give the key points. Let $\delta_{N,T} = \min(\sqrt{N}, T)$. Following Bai (2004),

$$
\hat{F}_{1,t} - H_{1}F_{t} = V_{NT}^{-\frac{1}{2}} \left( \frac{1}{T^2} \sum_{t=1}^{T} \hat{F}_{1,s} \xi_{st} + \frac{1}{T^2} \sum_{t=1}^{T} \hat{F}_{1,s} \eta_{st} + \frac{1}{T^2} \sum_{t=1}^{T} \hat{F}_{1,s} \xi_{st} + \frac{1}{T^2} \sum_{t=1}^{T} \hat{F}_{1,s} \gamma_{st} \right). \quad (7)
$$

Lemma 9. $\sum_{t=1}^{T} || \sum_{s=1}^{\tau} (\hat{F}_{1,t} - H_{1}F_{t})F'_{i} ||^{2} = O_{p}(\frac{T^{2}}{\min(N,TT^{2})})$.

Lemma 10. $\sum_{t=1}^{T} || \sum_{s=1}^{\tau} (\hat{F}_{1,t} - H_{1}F_{t})u_{it} ||^{2} = O_{p}(\frac{T^{3}}{\min(N,T,T^{2/3})})$ for each $i$.

Lemma 11. $\frac{1}{T^{2}} \sum_{t=1}^{T} (\hat{F}_{1,t} - H_{1}F_{t})F'_{i} = O_{p}(\frac{1}{T \min(\sqrt{N}, T)})$.

Lemma 12. $\frac{1}{T^{2}} \sum_{t=1}^{T} (\hat{F}_{1,t} - H_{1}F_{t})u_{it} = O_{p}(\frac{1}{T \min(\sqrt{N}, T)^{3/2}/T^{2}})$.

Lemma 13. $\frac{1}{T^{2}} \sum_{t=1}^{T} || \hat{F}_{1,t} - H_{1}F_{t} ||^{2} = O_{p}(\frac{T}{\min(N,T,T^{2})})$.

Lemma 14. $\frac{1}{T^{2}} \sum_{t=1}^{T} (\hat{F}_{1,t}F'_{i} - H_{1}F_{t}(H_{1}F'_{i})) = O_{p}(\frac{1}{T \min(\sqrt{N}, T)})$. Therefore, $\hat{\Sigma}_{p}^{-1} = \Sigma_{p}^{-1} + o_{p}(1)$.

Proof of Lemma 9

By (7),

$$
\sum_{t=1}^{\tau} || \sum_{s=1}^{\tau} (\hat{F}_{1,t} - H_{1}F_{t})F'_{i} ||^{2} \leq 4||V_{NT}^{-\frac{1}{2}}|| \left( \sum_{t=1}^{\tau} \left( \frac{1}{T^{2}} \sum_{t=1}^{T} \sum_{s=1}^{\tau} \hat{F}_{1,s} \xi_{st} F'_{i} \right)^{2} \right) + \sum_{t=1}^{\tau} \left( \frac{1}{T^{2}} \sum_{t=1}^{T} \sum_{s=1}^{\tau} \hat{F}_{1,s} \xi_{st} F'_{i} \right)^{2}
$$

$$
+ \sum_{t=1}^{\tau} \left( \frac{1}{T^{2}} \sum_{t=1}^{T} \sum_{s=1}^{\tau} \hat{F}_{1,s} \eta_{st} F'_{i} \right)^{2} + \sum_{t=1}^{\tau} \left( \frac{1}{T^{2}} \sum_{t=1}^{T} \sum_{s=1}^{\tau} \hat{F}_{1,s} \xi_{st} F'_{i} \right)^{2}.
$$

First consider part I. By adding and subtracting terms,

$$
\sum_{t=1}^{\tau} \left( \frac{1}{T^{2}} \sum_{t=1}^{T} \sum_{s=1}^{\tau} \hat{F}_{1,s} \gamma_{st} F'_{i} \right)^{2} \leq 2 \left( \frac{1}{T^{2}} \sum_{t=1}^{T} \sum_{s=1}^{\tau} (\hat{F}_{1,s} - H_{1}F_{s}) \gamma_{st} F'_{i} \right)^{2} + \frac{1}{T^{2}} \sum_{t=1}^{\tau} \sum_{s=1}^{\tau} H_{1}F_{s} \gamma_{st} F'_{i} \right)^{2}.
$$
For the first term,
\[
\frac{1}{T^4} \sum_{t=1}^{\tilde{T}} \left\| \sum_{j=1}^{\tau} \sum_{s=1}^{T} (\hat{F}_{t,s} - H'_j F_s) \gamma_{st} F'_t \right\|^2 \leq \frac{\tilde{T}}{T^4} \left( \frac{1}{T} \sum_{s=1}^{T} \left\| \hat{F}_{t,s} - H'_j F_s \right\|^2 \right) \left( \frac{1}{N} \sum_{j=1}^{N} E(u_{ij}^2) \right)^2 \left( \sum_{t=1}^{\tilde{T}} \left\| F'_t \right\|^2 \right)^2 \leq O_p(\frac{\tilde{T}^2}{T} \delta_{i,N_T}^{-2}),
\]
by Lemma B.1 in Bai (2004) and \( (\sum_{t=1}^{\tilde{T}} \left\| F'_t \right\|^2)^2 \leq T \sum_{t=1}^{\tilde{T}} \left\| F'_t \right\|^2 = O_p(TT^2) \).

Consider the second term,
\[
\frac{1}{T^4} \sum_{t=1}^{\tilde{T}} \left\| \sum_{j=1}^{\tau} \sum_{s=1}^{T} H'_j F_s \gamma_{st} F'_t \right\|^2 \leq \frac{\tilde{T}}{T^4} \left( \frac{1}{N} \sum_{j=1}^{N} E(u_{ij}^2) \right)^2 \left( \sum_{t=1}^{\tilde{T}} \left\| F'_t \right\|^2 \right)^2 \leq O_p(\tilde{T}),
\]
since \( \left\| H'_i \right\| = O_p(1) \) from Bai (2004). Therefore \( I \) is \( O_p(\tilde{T}) \).

Consider part \( II \).
\[
\sum_{t=1}^{\tilde{T}} \left( \frac{1}{T^2} \sum_{j=1}^{\tau} \sum_{s=1}^{T} (\hat{F}_{t,s} - H'_j F_s) \gamma_{st} F'_t \right)^2 \leq 2 \left( \frac{1}{T^2} \sum_{t=1}^{\tilde{T}} \left\| \sum_{i=1}^{T} (\hat{F}_{t,s} - H'_j F_s) \gamma_{st} F'_t \right\|^2 + \frac{1}{T^2} \sum_{t=1}^{\tilde{T}} \left\| \sum_{i=1}^{T} H'_j F_s \gamma_{st} F'_t \right\|^2 \right),
\]
where
\[
\frac{1}{T^4} \sum_{t=1}^{\tilde{T}} \left\| \sum_{j=1}^{\tau} \sum_{s=1}^{T} (\hat{F}_{t,s} - H'_j F_s) \gamma_{st} F'_t \right\|^2 \leq \frac{T}{N^2 \tilde{T}^3} \sum_{j=1}^{T} \left\| \hat{F}_{t,s} - H'_j F_s \right\|^2 \sum_{t=1}^{\tilde{T}} \left\| F'_t \right\|^2 \leq O_p(\frac{\tilde{T}^2}{N} \delta_{i,N_T}^{-2}),
\]
and
\[
\frac{1}{T^4} \sum_{t=1}^{\tilde{T}} \left\| \sum_{j=1}^{\tau} \sum_{s=1}^{T} H'_j F_s \gamma_{st} F'_t \right\|^2 \leq \frac{T}{N^2 \tilde{T}^3} \sum_{j=1}^{T} \left\| F'_t \right\|^2 \sum_{t=1}^{\tilde{T}} \sum_{j=1}^{T} \left\| \sum_{i=1}^{N} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (u_{ij}u_{it} - E(u_{ij}u_{it})) \right\|^2 \sum_{t=1}^{\tilde{T}} \left\| F'_t \right\|^2 \leq O_p(\frac{\tilde{T}^2}{N}).
\]
Thus, \( II \) is \( O_p(\frac{T^2}{N}) \).

Consider part \( III \).

\[
\sum_{\tau=1}^{\hat{T}} \left\| \frac{1}{T^2} \sum_{i=1}^{T} \sum_{s=1}^{T} (\hat{F}_{i,s} - H_i F_s) \eta_{st} F'_t \right\|^2 \leq 2 \left( \frac{1}{T^4} \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} (\hat{F}_{i,s} - H_i F_s) \eta_{st} F'_t \right\|^2 + \frac{1}{T^4} \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} H_i F_s \eta_{st} F'_t \right\|^2 \right).
\]

For the first expression in the bracket,

\[
\frac{1}{T^4} \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} \sum_{s=1}^{T} (\hat{F}_{i,s} - H_i F_s) \eta_{st} F'_t \right\|^2 \leq \frac{1}{N^2 T^3} \left( \frac{1}{T} \sum_{s=1}^{T} \left\| \hat{F}_{i,s} - H_i F_s \right\|^2 \right) \sum_{i=1}^{T} \left\| F'_t \right\|^2 \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} \eta_{st} F'_t \right\|^2
\]

\[
= O_p(\frac{TT^2}{N} \delta_{T,NT}^{-2})
\]

as \( \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} \eta_{st} F'_t \right\|^2 = O_p(NTT^2) \) by FCLT.

For the second expression,

\[
\frac{1}{T^4} \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} \sum_{s=1}^{T} H_i F_s \eta_{st} F'_t \right\|^2 \leq \frac{1}{N^2 T^4} \left( \sum_{i=1}^{T} \left\| F'_t \right\|^2 \right) \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} \eta_{st} F'_t \right\|^2 \left\| H_i F_s \right\|^2
\]

\[
= O_p(\frac{TT^2}{N}).
\]

As a result, \( III \) is \( O_p(\frac{T^2}{N}) \).

Finally consider part \( IV \).

\[
\sum_{\tau=1}^{\hat{T}} \left\| \frac{1}{T^2} \sum_{i=1}^{T} \sum_{s=1}^{T} (\hat{F}_{i,s} - H_i F_s) \xi_{st} F'_t \right\|^2 \leq 2 \left( \frac{1}{T^4} \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} (\hat{F}_{i,s} - H_i F_s) \xi_{st} F'_t \right\|^2 + \frac{1}{T^4} \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} H_i F_s \xi_{st} F'_t \right\|^2 \right).
\]

For the first term,

\[
\frac{1}{T^4} \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} \sum_{s=1}^{T} (\hat{F}_{i,s} - H_i F_s) \xi_{st} F'_t \right\|^2 \leq \frac{\hat{T}}{N^2 T^3} \left( \frac{1}{T} \sum_{s=1}^{T} \left\| \hat{F}_{i,s} - H_i F_s \right\|^2 \right) \sum_{i=1}^{T} \left\| \sum_{st} \eta_{st} F'_t \right\|^2 \sum_{\tau=1}^{\hat{T}} \left\| \sum_{i=1}^{T} \xi_{st} F'_t \right\|^2
\]

\[
= O_p(\frac{TT^2}{N} \delta_{T,NT}^{-2}).
\]
Consider the second term,

\[
\frac{1}{T^4} \sum_{\tau=1}^{\tilde{T}} \left\| \sum_{t=1}^{\tau} \sum_{s=1}^{T} H_{s}^{H} F_{s} \tilde{\xi}_{st} F_{t}^{H} \right\|^2 \leq \frac{T}{N^2 T} \| \sum_{s=1}^{T} F_{s} u_{s} A_{1} \|^{2} \left( \sum_{t=1}^{\tilde{T}} \| F_{t} \|^2 \right) \| H_{1} \|^2
\]

\[
= O_{p}(\frac{T T^2}{N^3}),
\]

because \( \| \sum_{s=1}^{T} F_{s} u_{s} A_{1} \| = O_{p}(\sqrt{NT}) \). Thus, IV is \( O_{p}(\frac{T T^2}{N^3}) \). As \( \| V_{NT} \| = O_{p}(1) \) from Bai (2004), the proof is finished by combining these results.

**Proof of Lemma 10**

\[
\sum_{\tau=1}^{\tilde{T}} \left\| \sum_{t=1}^{\tau} \sum_{s=1}^{T} (F_{s}^{H} - H_{s}^{H} F_{s}) u_{st} \right\|^2 \leq 4 \| V_{NT} \| \left( \sum_{\tau=1}^{\tilde{T}} \left\| \frac{1}{T^2} \sum_{t=1}^{\tau} \sum_{s=1}^{T} \tilde{F}_{s} \xi_{st} u_{st} \right\|^2 \right) + \sum_{\tau=1}^{\tilde{T}} \left\| \frac{1}{T^2} \sum_{t=1}^{\tau} \sum_{s=1}^{T} \tilde{F}_{s} \xi_{st} u_{st} \right\|^2
\]

\[
+ \sum_{\tau=1}^{\tilde{T}} \left\| \frac{1}{T^2} \sum_{t=1}^{\tau} \sum_{s=1}^{T} \hat{F}_{s} \xi_{st} u_{st} \right\|^2 + \sum_{\tau=1}^{\tilde{T}} \left\| \frac{1}{T^2} \sum_{t=1}^{\tau} \sum_{s=1}^{T} \hat{F}_{s} \xi_{st} u_{st} \right\|^2
\]

We begin with part I.

\[
\sum_{\tau=1}^{\tilde{T}} \left\| \frac{1}{T^2} \sum_{t=1}^{\tau} \sum_{s=1}^{T} \hat{F}_{s} \xi_{st} u_{st} \right\|^2 \leq 2 \left( \frac{1}{T^2} \sum_{\tau=1}^{\tilde{T}} \left\| \sum_{t=1}^{\tau} \sum_{s=1}^{T} (\hat{F}_{s} - H_{s} F_{s}) \xi_{st} u_{st} \right\|^2 + \frac{1}{T^2} \sum_{\tau=1}^{\tilde{T}} \left\| \sum_{t=1}^{\tau} \sum_{s=1}^{T} H_{s} F_{s} \xi_{st} u_{st} \right\|^2 \right).
\]

The first expression on the right hand side is bounded by

\[
\frac{T^2}{T^4} \left( \frac{1}{T} \sum_{s=1}^{T} \left\| \hat{F}_{s} - H_{s} F_{s} \right\|^2 \right) \left( \frac{1}{T} \sum_{\tau=1}^{\tilde{T}} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{\tau} u_{st} \frac{1}{N} \sum_{j=1}^{N} E(u_{st}^2) \right)^2 \right)
\]

which is \( O_{p}(\frac{T^2}{T^3} \delta^{-2}_{NT}) \). The second term is bounded by

\[
\frac{1}{T^4} \sum_{\tau=1}^{\tilde{T}} \left\| \sum_{t=1}^{\tau} \sum_{s=1}^{T} F_{s} \xi_{st} u_{st} \right\| \frac{1}{N} \sum_{j=1}^{N} E(u_{st}^2) ||H_{s}||^2 = O_{p}(\frac{T}{T^2}),
\]

by FCLT. Thus, I is \( O_{p}(\frac{T}{T^2}) \).
Now Part II.

\[
\sum_{t=1}^{T} \left| \frac{1}{T} \sum_{i=1}^{T} \sum_{s=1}^{T} \hat{F}_{i,s} \xi_{at} u_{it} \right|^2 \leq 2 \left( \frac{1}{T^2} \sum_{t=1}^{T} \left| \sum_{i=1}^{T} (\hat{F}_{i,s} - H_{i}^{*} F_{i}) \xi_{at} u_{it} \right|^2 + \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{i=1}^{T} H_{i}^{*} F_{i} \xi_{at} u_{it} \right|^2 \right).
\]

The first is similar to that in Lemma 2, which is \(O_{p}(\frac{T^3}{NT^2} \delta_{i,NT}^{-2})\). The second is \(O_{p}(\frac{T^3}{NT})\). Thus, II is \(O_{p}(\frac{T^3}{NT})\).

Consider part III.

\[
\sum_{t=1}^{T} \left| \frac{1}{T} \sum_{i=1}^{T} \sum_{s=1}^{T} \hat{F}_{i,s} \eta_{at} u_{it} \right|^2 \leq 2 \left( \frac{1}{T^2} \sum_{t=1}^{T} \left| \sum_{i=1}^{T} (\hat{F}_{i,s} - H_{i}^{*} F_{i}) \eta_{at} u_{it} \right|^2 + \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{i=1}^{T} H_{i}^{*} F_{i} \eta_{at} u_{it} \right|^2 \right).
\]

The first term is bounded by \(\frac{\bar{F}^2}{NT^2} (\frac{1}{T} \sum_{s=1}^{T} \left| (\hat{F}_{i,s} - H_{i}^{*} F_{i}) \sum_{i=1}^{T} \left| F_{i} \right|^2 \sum_{i=1}^{T} u_{it}^2 \sum_{i=1}^{T} \left| \frac{1}{\sqrt{T}} \lambda_{i} \right|^2)\), which is \(O_{p}(\frac{T^3}{NT})\). For the second term, similar to that in Lemma 2,

\[
\frac{1}{T^2} \sum_{t=1}^{T} \left| \sum_{i=1}^{T} \sum_{s=1}^{T} H_{i}^{*} F_{i} \eta_{at} u_{it} \right|^2 \leq O_{p}(\frac{\bar{F}^2}{N}) + O_{p}(\frac{T^3}{N^2}).
\]

III is thus \(O_{p}(\frac{T^3}{N \min(N,T)})\).

Finally part IV.

\[
\sum_{t=1}^{T} \left| \frac{1}{T} \sum_{i=1}^{T} \sum_{s=1}^{T} \hat{F}_{i,s} \xi_{at} u_{it} \right|^2 \leq 2 \left( \frac{1}{T^2} \sum_{t=1}^{T} \left| \sum_{i=1}^{T} (\hat{F}_{i,s} - H_{i}^{*} F_{i}) \xi_{at} u_{it} \right|^2 + \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{i=1}^{T} H_{i}^{*} F_{i} \xi_{at} u_{it} \right|^2 \right).
\]

For the first expression in the bracket,

\[
\frac{1}{T^2} \sum_{t=1}^{T} \left| \sum_{i=1}^{T} \sum_{s=1}^{T} (\hat{F}_{i,s} - H_{i}^{*} F_{i}) \xi_{at} u_{it} \right|^2 \leq \frac{1}{NT^2} \left( \frac{1}{T} \sum_{s=1}^{T} \left| H_{i}^{*} F_{i} \right|^2 \right) \left( \frac{1}{T} \sum_{s=1}^{T} \left| \frac{1}{\sqrt{T}} \lambda_{i} \right|^2 \right) \sum_{t=1}^{T} \left| F_{i} u_{it} \right|^2 = O_{p}(\frac{T}{N \delta_{i,NT}^{-2}}).
\]

The second term is bounded by \(\frac{1}{NT^2} \sum_{s=1}^{T} \left| H_{i}^{*} F_{i} \right|^2 \sum_{t=1}^{T} \left| F_{i} u_{it} \right|^2 \left| H_{i}^{*} \right|^2\), which is \(O_{p}(\frac{T}{\bar{F}})\). Thus, IV is \(O_{p}(\frac{T}{\bar{F}})\).

As a result, \(\sum_{t=1}^{T} \left| \sum_{i=1}^{T} (\hat{F}_{i,s} - H_{i}^{*} F_{i}) u_{it} \right|^2 = 4\left| \frac{1}{\sqrt{T}} \sum_{s=1}^{T} \left| \lambda_{i} \right|^2 \right| (I + II + III + IV) = O_{p}(\frac{T^3}{\min(N^2,NT,T^3)})\).
Proof of Lemmas 11-13

The proofs are straightforwardly adapted from Lemma B.4 of Bai (2004), Lemma B.1 of Bai (2003) and Theorem 1 of Bai & Ng (2002). The difference is mainly that the sum is taken over \( \tilde{T} \) periods in my case, but is taken over \( T \) periods in Bai & Ng (2002), Bai (2003) and Bai (2004). The adaptation is very similar to that in the proof of Lemma 3 and is thus omitted.

Proof of Lemma 14

It is similar to Lemma 6 and is thus omitted.

Proof of Theorem 3

As the proof is similar to that of Theorem 1, we start with (4) which now becomes

\[
\frac{1}{\tilde{T}T^2} \sum_{\tau=1}^{T-[T\pi]} \| \hat{\delta}_{\tau}(\pi) - \delta_{\tau}(\pi) \|^2 \leq \frac{3}{\tilde{T}T^2} \sum_{\tau=1}^{\tilde{T}} \| I \|^2 + \frac{3}{\tilde{T}T^2} \sum_{\tau=1}^{\tilde{T}} \| II \|^2 + \frac{3}{\tilde{T}T^2} \sum_{\tau=1}^{\tilde{T}} \| III \|^2.
\]

By Lemma 9 and 10, \( \sum_{\tau=1}^{\tilde{T}} \| I \|^2 = O_p(\frac{\tilde{T}T^2}{\min(N,T^2)}) \). By Lemma 11,12 and 14, \( \sum_{\tau=1}^{\tilde{T}} \| II \|^2 = O_p(\frac{\tilde{T}T^2}{\min(N,T^2)}) \).

By Lemma 10, \( \sum_{\tau=1}^{\tilde{T}} \| III \|^2 = O_p(\frac{\tilde{T}T^2}{\min(N,T^2)}) \). As a result, \( \hat{\text{LM}}_i(\pi) = \text{LM}_i(\pi) + O_p(\frac{1}{\sqrt{\min(N,T^2)}}) \).

D Dataset

We apply the following transformations to the raw data:

1. Take logs if Logs=1
2. Take the first difference if Diff=1 or the second difference if Diff=2;
3. Apply an one-sided moving average filter with a 3-period window if Filter=3 or with a 12-period window if Filter=12.
<table>
<thead>
<tr>
<th>Release</th>
<th>Series</th>
<th>Logs</th>
<th>Diff</th>
<th>Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>GFK consumer confidence</td>
<td>Aggregate balance</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>GFK consumer confidence</td>
<td>How does the financial situation of your household now compare with what it was 12 months ago</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>GFK consumer confidence</td>
<td>How do you think the financial position of your household will change over the next 12 months</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>GFK consumer confidence</td>
<td>How do you think the general economic situation in this country has changed over the last 12 months</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>GFK consumer confidence</td>
<td>How do you think the general economic situation in this country will develop over the next 12 months</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>GFK consumer confidence</td>
<td>How do you think the level of unemployment will change over the next 12 months?</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>GFK consumer confidence</td>
<td>Do you think that there is an advantage for people to make major purchases at the present time</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>GFK consumer confidence</td>
<td>Over the next 12 months how do you think the amount of money you’ll spend on major purchases will compare with what you spent over the last 12 months</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS retail sale</td>
<td>All Retailers (Volume seasonally adjusted)</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS retail sale</td>
<td>All Retailers (Value seasonally adjusted)</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS retail sale</td>
<td>Predominantly food stores (Volume seasonally adjusted)</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS retail sale</td>
<td>Predominantly food stores (Value seasonally adjusted)</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS retail sale</td>
<td>Non-specialised stores</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS retail sale</td>
<td>Textiles: clothing: footwear</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS retail sale</td>
<td>Household goods stores</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS retail sale</td>
<td>Non-store retailing &amp; repair</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Retailing Sales</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Retailing Orders</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Retailing Sales for Time of Year</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Retailing Stocks</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Wholesaling Sales</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Wholesaling Orders</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Wholesaling Sales for Time of Year</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Motor Traders Sales</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Motor Traders Orders</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Motor Traders Sales for Time of Year</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Release</td>
<td>Series</td>
<td>Logs</td>
<td>Diff</td>
<td>Filter</td>
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<td>----------------------------------------------------</td>
<td>------</td>
<td>------</td>
<td>--------</td>
</tr>
<tr>
<td>CBI distributive trades</td>
<td>Motor Traders Stocks</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS travel</td>
<td>UK visits abroad: Expenditure abroad</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS travel</td>
<td>OS visits to UK: Earnings</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS trade</td>
<td>BOP: Balance, sa, Total Trade in Goods</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS trade</td>
<td>BOP: Balance, Manufactures</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS trade</td>
<td>BOP: Balance, Intermediate goods</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS trade</td>
<td>BOP: IM: CVM: sa: Total Trade in Goods</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS trade</td>
<td>BOP: EX: CVM: sa: Total Trade in Goods</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS trade</td>
<td>BOP: Balance, Capital goods</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS trade</td>
<td>BOP: IM, price index, Finished manufactures</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>ONS trade</td>
<td>Balance of Payments: Trade in Services: Total balance: CP</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>FTSE</td>
<td>All Share Dividend Yield</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>FTSE</td>
<td>All Share Price / Earnings Ratio</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>FTSE</td>
<td>All Share Price Index</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>FTSE</td>
<td>FTSE 100</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Exchange rate</td>
<td>Japanese Yen /£</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Exchange rate</td>
<td>United States Dollar /£</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Exchange rate</td>
<td>Effective exchange rate index, Sterling (Jan 2005=100)</td>
<td>1</td>
<td>1</td>
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<td>Bank Of England Repo Rate</td>
<td>0</td>
<td>1</td>
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<td>Interest rate</td>
<td>Overnight £ Inter-Bank Rate (Mean Libid/Libor)−8.30am</td>
<td>0</td>
<td>1</td>
<td>3</td>
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<td>Interest rate</td>
<td>ICE LIBOR GBP 3 Month</td>
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<td>3 Month £ Inter-Bank Rate (Mean Libid/Libor)−10.30am</td>
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<td>Interest rate</td>
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<td>1</td>
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<td>End month level of discount rate, 3 month Treasury bills</td>
<td>0</td>
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<tr>
<td>Volatility</td>
<td>LIFFE FTSE 100 3 Months Constant Maturity: Implied Volatility</td>
<td>0</td>
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<td>VRPSPOT(NOM,UK,5)</td>
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<td>AEI (including bonuses), whole economy</td>
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<td>AEI (including bonuses), private sector</td>
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<td>In employment: UK: All: Aged 16+</td>
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<td>Unemployed: UK: All: Aged 16+</td>
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<td>Total actual weekly hours worked (millions): UK: All</td>
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<td>Unemployed up to 6 months: UK: All: Aged 16+</td>
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<td>Unemployed over 6 and up to 12 months: UK: All: Aged 16+</td>
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<td>Unemployed over 12 months: UK: All: Aged 16+</td>
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<td>MST</td>
<td>Monthly amounts outstanding of monetary financial institutions’ sterling M4 liabilities to private sector</td>
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<td>CIPS manufacturing</td>
<td>Consumer Goods Industries - Total New Orders</td>
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<td>1</td>
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<td>Supplier’s Delivery Times</td>
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<td>CIPS manufacturing</td>
<td>Total New Orders</td>
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<td>Output</td>
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<td>Stocks Of Purchases</td>
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<td>ONS output</td>
<td>Industry C: Mining &amp; quarrying</td>
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<td>ONS output</td>
<td>Industry D: Manufacturing</td>
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<td>Industry E: Electricity, gas and water supply</td>
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<td>Industry DB: Manuf of textile &amp; textile products</td>
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<td>ONS output</td>
<td>Industry DC: Manuf of leather &amp; leather products</td>
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<td>ONS output</td>
<td>Industry DD: Manuf of wood &amp; wood products</td>
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<td>ONS output</td>
<td>Industry DE: Pulp/paper/printing/publishing industries</td>
<td>1</td>
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<td>Industry DF: Manuf coke/petroleum prod/nuclear fuels</td>
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<td>ONS output</td>
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<td>ONS output</td>
<td>Industry DH: Manuf of rubber &amp; plastic products</td>
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<td>ONS output</td>
<td>Industry DI: Manuf of non-metallic mineral products</td>
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<td>1</td>
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<tr>
<td>ONS output</td>
<td>Industry DJ: Manuf of basic metals &amp; fabricated prod</td>
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<td>1</td>
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<td>ONS output</td>
<td>Industry DL: Manuf of electrical &amp; optical equipment</td>
<td>1</td>
<td>1</td>
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<td>ONS output</td>
<td>Industry DM: Manuf of transport equipment</td>
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<td>1</td>
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<td>CBI monthly trends</td>
<td>Do you consider that in volume terms, your present total order book is above normal?</td>
<td>0</td>
<td>1</td>
<td>3</td>
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<td>CBI monthly trends</td>
<td>Do you consider that in volume terms, your present export order book is above normal?</td>
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<td>3</td>
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<td>CBI monthly trends</td>
<td>Adequacy of Stocks of Finished Goods</td>
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<td>1</td>
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<td>What is the expected trend over the next 4 months with regards to your volume of output?</td>
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<td>What is the expected trend over the next 4 months with regards to average prices for domestic orders?</td>
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<td>Brent crude</td>
<td>1 Month Fwd, fob US/BBL</td>
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<td>Physical Del., fob US/BBL</td>
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<td>ONS PPI</td>
<td>NSI: All manufacturing: Materials only</td>
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<td>2</td>
<td>3</td>
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<td>ONS PPI</td>
<td>Prod of man ind excl.f.b, p &amp; t sa</td>
<td>1</td>
<td>2</td>
<td>3</td>
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<td>ONS PPI</td>
<td>Fuels Purchased by Man Ind Excl CCL</td>
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<td>2</td>
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<td>NSI: M &amp; F purchased by Man: Excl FBPT Excl CCL NSA</td>
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<td>ONS PPI</td>
<td>Output of manufactured products</td>
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<td>3</td>
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<td>ONS PPI</td>
<td>NSO: All Manufacturing excl duty: sa</td>
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<td>2</td>
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<tr>
<td>ONS CPI</td>
<td>Food And Non-Alcoholic Beverages</td>
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<td>2</td>
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<td>Alcoholic Beverages, Tobacco &amp; Narcotics</td>
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<td>2</td>
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<td>Clothing And Footwear</td>
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<td>2</td>
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<td>2</td>
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<td>2</td>
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<td>12</td>
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<td>2</td>
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<td>Hotels, Cafes And Restaurants</td>
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<td>12</td>
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